

Calculating Limits

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This note concerns the purely mechanical calculation of some of the most common types of ordinary limits one sees in an elementary Calculus course. In a nutshell, the calculation of a limit which is indeterminate is generally done by simplifying the original function to obtain a new function which is equal to the original function except at the limit point and for which the limit is obvious.

This may be expressed as follows. One tries to calculate $\lim_{x \rightarrow c} f(x)$ but finds it is indeterminate. One simplifies the formula defining f and finds a new function g such that $f(x) = g(x)$ when $x \neq c$ and finds $\lim_{x \rightarrow c} g(x) = L$. One may then conclude $\lim_{x \rightarrow c} f(x) = L$.

Frequently, the simplifications involved in finding g fall into one of three categories.

Category 1: The limit involves a quotient of polynomials. Often, $f(x)$ is a quotient of polynomials, both of which are equal to 0 when $x = c$. In this case, one can always factor $x - c$ from both the numerator and denominator and cancel (when $x \neq c$) to obtain a formula for a new function equal to $f(x)$ for $x \neq c$.

Example: $\lim_{x \rightarrow 4} \frac{x^2 + 3x - 28}{x^2 - x - 12}$. One may factor the numerator and denominator to obtain

$$\lim_{x \rightarrow 4} \frac{x^2 + 3x - 28}{x^2 - x - 12} = \lim_{x \rightarrow 4} \frac{(x - 4)(x + 7)}{(x - 4)(x + 3)} = \lim_{x \rightarrow 4} \frac{x + 7}{x + 3} = \frac{11}{7}.$$

Category 2: The limit involves a radical. Often, $f(x)$ will involve a quotient where the numerator contains a radical. In this case, one simplifies by *rationalizing* the numerator.

Example: $\lim_{x \rightarrow 49} \frac{\sqrt{x} - 7}{x - 49}$. One may rationalize the numerator to obtain

$$\lim_{x \rightarrow 49} \frac{\sqrt{x} - 7}{x - 49} = \lim_{x \rightarrow 49} \frac{(\sqrt{x} - 7)(\sqrt{x} + 7)}{(x - 49)(\sqrt{x} + 7)} = \lim_{x \rightarrow 49} \frac{x - 49}{(x - 49)(\sqrt{x} + 7)} =$$
$$\lim_{x \rightarrow 49} \frac{1}{\sqrt{x} + 7} = \frac{1}{14}.$$

Category 3: The limit involves a complex rational expression. Often, $f(x)$ is a quotient where the numerator itself is a rational expression rather than a polynomial. In this case, one may simplify by multiplying the numerator and the denominator by the least common denominator of the terms in the numerator.

Example: $\lim_{x \rightarrow 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$. Here, the least common denominator of the terms in the numerator, $\frac{1}{x}$ and $\frac{1}{3}$, is $3x$, so one may multiply the numerator and denominator by $3x$.

$$\lim_{x \rightarrow 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3} = \lim_{x \rightarrow 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3} \cdot \frac{3x}{3x} = \lim_{x \rightarrow 3} \frac{3 - x}{(3x)(x - 3)} = \lim_{x \rightarrow 3} \frac{-1}{3x} = -\frac{1}{9}.$$