

Analyzing the Sign of a Function

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In Calculus, we see that many problems involving change can be dealt with by analyzing the sign of a derivative. This makes the ability to analyze the sign of a function (remember, among other things, a derivative is itself a function) extremely important (no pun intended). Indeed, hardly a day will go by in which we will not have to do such an analysis several times.

Analyzing the sign of a function is equivalent to solving an inequality. For example x is in the solution set of the inequality $f(x) < g(x)$ if and only if $f(x) - g(x)$ is negative. Thus, any method that one can use to solve an inequality can be used to analyze the sign of a function. The method described here is probably the easiest method to use in a large proportion of cases. We will describe the method for the case where the function involved is a rational function, i.e. a quotient of polynomial functions, but the basic idea can be used with more general functions.

Laying the Groundwork with Linear Functions. First, consider linear functions, that is, functions of the form $f(x) = ax + b$ for constants a, b . Actually, since $ax + b = a(x + b/a)$, we really only have to consider functions of the form $f(x) = x + c$ or $f(x) = x - c$. The key fact is that it's very easy to determine the sign of $f(x) = x + c$ on any interval not containing $-c$, and it's very easy to determine the sign of $f(x) = x - c$ on any interval not containing c . Consider the following examples.

- (1) Determine the sign of $x - 3$ on the interval $(5, \infty) = \{x : x > 5\}$.
Analysis: If $x > 5$, then $x - 3 > 5 - 3$, so $x - 3 > 2$ and thus $x - 3$ is positive on $(5, \infty)$.
- (2) Determine the sign of $x - 3$ on the interval $(5, 10) = \{x : 5 < x < 10\}$.
Analysis: If $5 < x < 10$, then $5 - 3 < x - 3 < 10 - 3$, so $2 < x - 3 < 7$ and thus $x - 3$ is positive on $(5, 10)$.
- (3) Determine the sign of $x - 3$ on the interval $(3, \infty) = \{x : x > 3\}$.
Analysis: If $x > 3$, then $x - 3 > 3 - 3$, so $x - 3 > 0$ and thus $x - 3$ is positive on $(3, \infty)$.
- (4) Determine the sign of $x - 3$ on the interval $(-\infty, 3) = \{x : x < 3\}$.
Analysis: If $x < 3$, then $x - 3 < 3 - 3$, so $x - 3 < 0$ and thus $x - 3$ is negative on $(-\infty, 3)$.
- (5) Determine the sign of $x + 3$ on the interval $(-\infty, -3) = \{x : x < -3\}$.
Analysis: If $x < -3$, then $x + 3 < -3 + 3$, so $x + 3 < 0$ and thus $x + 3$ is negative on $(-\infty, -3)$.
- (6) Determine the sign of $x + 3$ on the interval $(5, 10) = \{x : 5 < x < 10\}$.
Analysis: If $5 < x < 10$, then $5 + 3 < x + 3 < 10 + 3$, so $8 < x + 3 < 13$ and thus $x + 3$ is positive on $(5, 10)$.

Look for the general idea: To analyze the sign of $x \pm c$ on an interval (α, β) , take the inequality $\alpha < x < \beta$, add or subtract c to all sides to get $\alpha \pm c < x \pm c < \beta \pm c$ and draw the obvious conclusion about the sign of $x \pm c$.

This suggests the following general strategy for analyzing the sign of more general functions.

Strategy.

- (1) Factor the function as much as possible. If the function involves a quotient, factor both the numerator and denominator as much as possible. In the best of all possible worlds, each factor will be a linear function.
- (2) Find all the zeroes of each of the factors of both the numerator and the denominator.
- (3) Mark off those points on a number line. These points partition the real line into a set of intervals.
- (4) Determine the sign of the function on each of those intervals. This may be done by examining the sign of each of the factors of the function. If the factors are linear functions, then the method described earlier will work.

Example. Consider the function $f(x) = \frac{5(x-3)(x+8)}{x(x-1)^2}$. Note that this function is completely factored. If the function you are dealing with is not factored, you will need to factor it completely.

One can tell at sight that $x - 3 = 0$ when x is 3, that $x + 8 = 0$ when x is -8 , that $x = 0$ when x is 0 (that didn't take much thought, did it?), and that $x - 1 = 0$ when x is 1. So set up and mark off a number line as shown below.

Now take a look at each interval, one at a time.

- (1) $(3, \infty)$
If $x > 3$, each factor is clearly positive, so f is positive on this interval.
- (2) $(1, 3)$
If $1 < x < 3$, then $x - 3$ is negative, since $x < 3$, but every other factor is positive, so f is negative on this interval.
- (3) $(0, 1)$
If $0 < x < 1$, then $x - 3$ is negative. Although $x - 1$ is negative, $(x - 1)^2$ is positive, since it's a square of a non-zero number, and every other factor is positive, so f is negative on this interval.
- (4) $(-8, 0)$
If $-8 < x < 0$, then $x - 3$ is negative, $x + 8$ is positive (since $x > -8$), but x is negative while $(x - 1)^2$ is positive, so f is positive on this interval.
- (5) $(-\infty, -8)$
If $x < -8$, then $x - 3$ is negative, $x + 8$ is negative (since $x < -8$), and x is negative, but $(x - 1)^2$ is positive, so f is negative on this interval.

We may visualize all this as follows.

If we wanted to express this result in standard mathematical notation, *as we always should want to*, we might say that f is positive on $(-8, 0) \cup (3, \infty)$, and f is negative on $(-\infty, -8) \cup (0, 1) \cup (1, 3)$. Alternatively, we might say that f is positive on $\{x : -8 < x < 0 \text{ or } x > 3\}$ and f is negative on $\{x : x < -8 \text{ or } 0 < x < 1 \text{ or } 1 < x < 3\}$.

The Theoretical Basis for the Method. The *Zero Theorem* tells us that if a function is continuous on a closed interval but has different signs at the endpoints of that interval it must be zero somewhere in between. It follows that a function must maintain the same sign at all points of an interval which does not contain any zeroes or discontinuities. If one factors the numerator and denominator of a function completely, the zeroes of the function will occur at the points where factors of the numerator are zero, while the discontinuities will generally occur where factors of the denominator are zero.