# Improper Integrals

There are basically two types of problems that lead us to define improper integrals.

(1) We may, for some reason, want to define an integral on an interval extending to $\pm\infty$. This leads to what is sometimes called an *Improper Integral of Type 1.*

(2) The integrand may fail to be defined, or fail to be continuous, at a point in the interval of integration, typically an endpoint. This leads to what is sometimes called an em Improper Integral of Type 2.

**Definition 1** (Improper Integral of Type 1)**.** (1) $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$ *provided the latter limit exists.*

(2) $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$ *provided the latter limit exists.*

**Definition 2** (Improper Integral of Type 2)**.** (1) *If $f$ is continuous on $[a,b)$ but discontinuous at $b$, then $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$ provided the latter limit exists.*

(2) *If $f$ is continuous on $(a,b]$ but discontinuous at $a$, then $\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$ provided the latter limit exists.*

If there's a problem at more than one point, we simply divide the integral up into a sum of integrals where each individual integral is of one of the types described.

If an improper integral is defined, we say it is convergent; otherwise we say the improper integral is divergent.

Sometimes, we are able to use the definition to determine whether an improper integral converges and, if so, what it converges to. Other times, we may need to use other means to determine whether it converges and approximate what it converges to.

If $f(x) \geq 0$ and its integral converges, we sometimes write that its integral is $< \infty$. Similarly, if $f(x) \geq 0$ and its integral diverges, we sometimes write that its integral $= \infty$.

The key tools are the *Comparison Test* and the *P-Test.*

**Theorem 1** (Comparison Test)**.** *Suppose $0 \leq f(x) \leq g(x)$ for $x \geq a$ and $\int_a^b f(x)\,dx$ exists for all $b > a$.*

(1) *If $\int_a^\infty g(x)\,dx < \infty$ then $\int_a^\infty f(x)\,dx < \infty$.*

(2) *If $\int_a^\infty f(x)\,dx = \infty$ then $\int_a^\infty g(x)\,dx = \infty$.*

Analogous tests work for each of the other types of improper integrals.

The Comparison Test suggests that, to examine the convergence of a given improper integral, we may be able to examine the convergence of a similar integral. To use it, we need a toolbox of improper integrals we know more about. The primary tool in that toolbox is the set of integrals of power functions.

**Theorem 2** (P-Test)**.** (1) $\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} < \infty & \text{for } p > 1 \\ = \infty & \text{for } p \leq 1 \end{cases}$

(2) $\int_0^1 \frac{1}{x^p}\,dx \begin{cases} = \infty & \text{for } p \geq 1 \\ < \infty & \text{for } p < 1 \end{cases}$

The P-Test is easy to verify. It may also be generalized very easily. The first case can actually be used for any interval of the form $[a,\infty)$ for $a > 0$, rather than just for $[1,\infty)$.

For the second case, it can be shown just as easily that, for any $a < b$,

(1) $\int_a^b \frac{1}{(x-a)^p}\,dx \begin{cases} = \infty & \text{for } p \geq 1 \\ < \infty & \text{for } p < 1 \end{cases}$

(2) $\int_a^b \frac{1}{(b-x)^p}\,dx \begin{cases} = \infty & \text{for } p \geq 1 \\ < \infty & \text{for } p < 1 \end{cases}$

For example, $\int_2^7 \frac{1}{\sqrt{x-2}}\,dx < \infty$ while $\int_{-5}^8 \frac{1}{(8-x)^3}\,dx = \infty$.

There are also variations of the Comparison Test, such as the following.

**Theorem 3** (Variation of the Comparison Test)**.** *Suppose $f(x)$ and $g(x)$ are both positive $x > x_0$ for some $x_0 \in \mathbb{R}$, both $\int_a^t f(x)\,dx$ and $\int_a^t g(x)\,dx$ exist for all $t > a$ and there exist $\alpha, \beta > 0$ such that $\alpha f(x) \leq g(x) \leq \beta g(x)$. Then either $\int_a^\infty f(x)\,dx$ and $int_a^\infty g(x)\,dx$ both converge or both diverge.*

There is an analogous variation for each of the types of improper integrals. There are in a sense redundant, since any integral that can be shown to converge using the variation could be shown to converge using the basic Comparison Test. However, they can be convenient and yield the following useful corollary.

**Corollary 4.** *Suppose $f(x)$ is continuous on $(a, b]$ and $\lim_{x\to a^+}(x-a)^p f(x) = c \neq 0$. Then* $\int_a^b f(x)\,dx \begin{cases} \textit{diverges} & \textit{if } p \geq 1 \\ \textit{converges} & \textit{if } p < 1 \end{cases}$

For example, $\int_2^5 \frac{1}{(x-2)(x+3)}\,dx$ is immediately seen to diverge, while $\int_2^5 \frac{1}{(x+3)\sqrt{x-2}}\,dx$ is immediately seen to converge.