Implicit Differentiation

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Why Do We Need Implicit Differentiation?

Despite many rumors to the contrary, not every function is defined explicitly by a formula of the form $y = f(x)$. Some functions are defined *implicitly* by equations of the form

$$g(x, y) = h(x, y), \tag{1}$$

where $g(x, y)$ and $h(x, y)$ represent mathematical expressions involving two variables. If, for every value of x in some set there is precisely one value of y such that $g(x, y) = h(x, y)$, and if we let $f(x) = y$, then f is a function, even though we have not written down a formula in the form $f(x) =$ something or other.

We have already seen several reasons why it is important to be able to calculate derivatives of functions. These reasons remain valid even if we do not have an explicit formula for a function. The method of implicit differentiation enables us to calculate the derivatives of such functions.

Example. Consider the equation

$$x^2y + y^2 = 5x + 11. \tag{2}$$

This equation has a graph, even though it may be beyond either our desire or capability to sketch it. Although the graph itself may not satisfy the infamous "vertical line test," and may not actually be the graph of a function, certainly we can erase enough of it so that what is left *is* the graph of a function. That function has been defined implicitly through (2).

The Method

The method we use for calculating the derivatives of such functions is actually relatively easy, *provided that we are familiar with the rules for calculating derivatives and have gotten into the habit of using the standard notation*. It consists of just two steps.

(1) Start with the equation (2), calculate the derivatives of each side with respect to the independent variable (in this case, x) using the ordinary, well-known rules for calculating derivatives and being careful to recognize the difference between an independent variable (in this case, x) and a dependent variable (in this case, y). This yields a new equation,

$$\frac{d}{dx}\left(g(x, y)\right) = \frac{d}{dx}\left(h(x, y)\right). \tag{3}$$

(2) If we have performed the first step correctly, then we can view (3) as an ordinary linear equation where the unknown we want to solve for is $\frac{dy}{dx}$. All we have to do now is solve that ordinary linear equation.

There is a drawback to this method, in that the formula we get depends not only on the independent variable, but on the dependent variable as well. As a result, it is only useful for calculating the derivative at points where we have a way of actually calculating the value of the function.

Example—Continued. Let us view equation (1) as defining a function $y = f(x)$ and try to calculate its derivative. (The derivative may be denoted by $f'(x)$, by $\frac{dy}{dx}$, by y', or by any number of other notations. When performing implicit differentiation, it is usually most convenient to use Leibniz' notation, so we will denote it by $\frac{dy}{dx}$.)

We start by writing

$$\frac{d}{dx}\left(x^2y + y^2\right) = \frac{d}{dx}\left(5x + 11\right). \tag{4}$$

If we calculate both derivatives correctly, we obtain

$$x^2\frac{dy}{dx} + y \cdot 2x + 2y \cdot \frac{dy}{dx} = 5. \tag{5}$$

Remembering that we want to treat (5) as a linear equation with $\frac{dy}{dx}$ as the unknown, we move each term involving $\frac{dy}{dx}$ to the left, and every other term to the right, obtaining

$$x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 5 - 2xy. \tag{6}$$

If we then factor $\frac{dy}{dx}$ from the left side of (6), we obtain

$$(x^2 + 2y)\frac{dy}{dx} = 5 - 2xy, \tag{7}$$

which is easily solved by dividing both sides by the coefficient $x^2 + 2y$ of the unknown $\frac{dy}{dx}$, obtaining

$$\frac{dy}{dx} = \frac{5 - 2xy}{x^2 + 2y}. \tag{8}$$

Suppose we want to use (8) to get an equation for the line tangent to the graph of (2) at the point $(2, 3)$. Since (2) can be viewed as defining a function $y = f(x)$ with $f(2) = 3$, we can use (8) to evaluate $f'(2)$ as follows, recognizing that this will tell us the slope of the line we're after.

$$f'(2) = \left.\frac{dy}{dx}\right|_{(2,3)} = \frac{2 - 2 \cdot 2 \cdot 3}{2^2 + 2 \cdot 3} = -7/10. \tag{9}$$

Thus, the slope of the tangent line is $-7/10$. Since it goes through the point $(2, 3)$, we can use the point-slope formula to obtain the equation

$$y - 3 = (-7/10) \cdot (x - 2) \tag{10}$$

for the tangent line we're looking for.