Monotonicity, Extrema, Concavity and Sketching Graphs

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Introduction

The ability to sketch a graph of a function is extremely useful because it enables one to visualize the behavior of the function, thus giving a concrete meaning to the function and making it more than an abstraction. The sketching of the graph of a function is intimately tied to the study of monotonicity (increasing or decreasing), extrema and concavity, which in turn are tied to the first and second derivatives. (In what follows, we assume that the first and second derivatives are defined wherever relevant. If there are any points where either derivative is not defined, then such points need to be dealt with using algebraic techniques. In practice, such points stick out like sore thumbs and the behavior of a function near such points is usually obvious if one uses a little common sense.)

A Word of Advice

These notes will appear meaningless unless you visualize what you are reading. Therefore, read them with a pencil in your hand and use it. By the time you have finished, you should have drawn dozens of pictures.

Conventions and Notation

We will refer to a function $y = f(x)$. That means that the name of the function is f, there is an independent variable called x and a dependent variable called y. We will also refer to the horizontal axis as the $x-$axis and the vertical axis as the $y-$axis. In actual examples, the names may be changed to protect the innocent.

Monotonicity

Definition 1 (Monotonicity). *A function f is increasing on an interval I if $f(a) < f(b)$ whenever $a, b \in I, a < b$. A function f is decreasing on an interval I if $f(a) > f(b)$ whenever $a, b \in I, a < b$.*

Informally, if the graph of a function rises as we look from the left to the right ("go from left to right," since the world seems to be ruled by righties), then we say that the function is increasing. Similarly, if the graph falls as we go from left to right, then we say that the function is decreasing. We sometimes add the adjective "strictly" or use the words non-decreasing or non-increasing when we need to be particular, but usually we do not have to be too precise as long as we're consistent.

Often, just knowing where a function is increasing and where it is decreasing will give us enough information to draw a rough sketch. So, the first thing we try to find out when we want to sketch a graph is to find out where the function we are graphing is increasing and where it is decreasing. In order to do this, we find the use of the first derivative helpful.

Theorem 1 (Monotonicity)**.** *Consider a function f which is differentiable on an interval I. If $f'(x)$ is positive for all $x \in I$, then f is increasing on I. If $f'(x)$ is negative for all $x \in I$, then f is decreasing on I.*

The key ideas are that *the derivative represents rate of change* and *a function which is increasing is changing in a positive direction.* Therefore, a function will be increasing where its derivative is positive and decreasing where its derivative is negative. This leads to the following first step.

Step 1: Calculate $f'(x)$. Determine where (on which intervals) f' is positive and where it is negative. This will tell you where f is increasing and where it is decreasing.

The endpoints of such intervals are clearly important points which should be placed appropriately on the graph. This leads to the next two steps.

Step 2: Find the coordinates of the endpoints of the intervals determined in Step 1. Since the first coordinates have been found in Step 1, one merely has to use the formula for f to find the second coordinates.

Step 3: Place the points obtained in Step 2 on the graph and connect them appropriately. (This may need to be redone after you obtain more information.)

Extrema

Definition 2 (Extrema)**.** *A function f has a local maximum at α if there is an open interval I containing α such that if $x \in I, x \neq \alpha$, then $f(x) < f(a)$. A function f has a local minimum at α if there is an open interval I containing α such that if $x \in I, x \neq \alpha$, then $f(x) > f(a)$.*

Monotonicity is tied together with extrema. (Recall that a minimum or maximum is also referred to as an extremum.) Usually, if a function increases and then decreases it will have a maximum at the common endpoint of the two intervals. After performing Steps 1-3, any maximum or minimum will be apparent with a glance at what has been drawn. As a double check, either of the following two theorems can be used.

Theorem 2 (First Derivative Test for Extrema)**.** *Consider a function f which is differentiable on an open interval I containing the point α. If $f'(x) > 0$ for $x \in I, x < \alpha$, while $f'(x) < 0$ for $x \in I, x > \alpha$, then f has a local maximum at α. If $f'(x) < 0$ for $x \in I, x < \alpha$, while $f'(x) > 0$ for $x \in I, x > \alpha$, then f has a local minimum at α.*

Concavity

A graph is generally concave down near a minimum and concave up near a maximum. Knowing where a graph is concave down and where it is concave up further helps us to sketch a graph.

Theorem 3 (Concavity)**.** *If $f''(x) > 0$ for all x in some interval, then the graph of f is concave up on that interval. If $f''(x) < 0$ for all x in some interval, then the graph of f is concave down on that interval.*

Thus we can determine concavity by examining the second derivative. We can understand why by glancing at a portion of a graph which is concave up. As one looks left to right (the standard direction), the slope of the tangent appears to increase. Thus the derivative increases also, which leads to the expectation that the second derivative (f'') is positive.

The connection between the second derivative and concavity leads to the next series of steps.

Step 4: Calculate f'' and determine where it is positive and where it is negative. That tells us where the graph is concave up and where it is concave down.

The endpoints of such intervals are clearly important points. They are called points of inflection and should be placed appropriately on the graph. Thus the next two steps are reminiscent of Steps 2 and 3.

Step 5: Find the coordinates of the endpoints of the intervals determined in Step 4. Since the first coordinates have already been found, one merely has to use the formula for f to find the second coordinates.

Step 6: Place the points obtained in Step 5 on the graph and connect them appropriately.

Other Things To Consider

Often, an examination of monotonicity and concavity will give enough information to sketch a graph. Sometimes, it is useful to look at other properties such as intercepts, symmetry and asymptotes. These are described below. As you gain experience sketching graphs, you will begin to get a feeling for when it is worthwhile to give these properties more than a cursory consideration.

Intercepts. A point where a curve crosses one of the axes is called an intercept. There are two types of intercepts, $x-$intercepts and $y-$intercepts.

Since a curve crosses the $y-$axis at a point where the first coordinate equals 0, you can calculate the $y-$intercept by simply evaluating $f(0)$. This is usually fairly easy to calculate.

Since a curve crosses the $y-$axis at a point where the second coordinate equals 0, you can calculate the $x-$intercept by solving the equation $f(x) = 0$. Depending on the formula for f, this may be very difficult to solve and often is not worth the effort.

It's generally important to find the $y-$intercept when it's unclear whether the curve crosses the $y-$axis above or below the origin, since drawing a picture with the curve crossing the $y-$axis on the wrong side of the origin presents a misleading picture.

It's important to find the $x-$intercepts when you're not sure whether or not the curve crosses the $x-$axis.

Symmetry. There are two types of symmetry that are sometimes worth paying attention to, symmetry about the $y-$axis and symmetry about the origin.

Even functions are symmetric about the $y-$axis. You can check whether a function is even by seeing if $f(-x) = f(x)$ for all values of x. Polynomial functions that contain only even powers, such as $f(x) = x^8 - 5x^2 + 3$, are examples of even functions. The cosine function is also an example of an even function.

Odd functions are symmetric about the origin. You can check whether a function is odd by seeing if $f(-x) = -f(x)$ for all values of x. Polynomial functions that contain only odd powers, such as $f(x) = 10x^7 + 8x^3 - x$, are examples of odd functions. The sine function is also an example of an odd function.

It is never actually absolutely necessary to check for symmetry. However, if you recognize from the formula for a function that its graph should exhibit symmetry, then you have another check for whether you have sketched the graph correctly.

Asymptotes. There are two types of asymptotes, horizontal and vertical.

A graph will have a horizontal asymptote $y = \alpha$ if $\lim_{x\to\infty} f(x) = \alpha$. In that case, the right side of the curve will get closer and closer to the horizontal asymptote, the line $y = \alpha$. For example, if $f(x) = (10x + 3)/(2x - 1)$, then $\lim_{x\to\infty} f(x) = 5$, so that the line $y = 5$ is a horizontal asymptote.

Similarly, a graph will have a horizontal asymptote $y = \alpha$ if $\lim_{x\to-\infty} f(x) = \alpha$. In that case, the left side of the curve will get closer and closer to the horizontal asymptote, the line $y = \alpha$. For example, if $f(x) = (15x + 3)/(5x - 1)$, then $\lim_{x\to-\infty} f(x) = 3$, so that the line $y = 3$ is a horizontal asymptote.

It's worth checking for a horizontal asymptote on the right hand side if, for large x, either f is increasing and concave down or decreasing and concave up. (Clearly, there can be no such asymptote if f is increasing and concave up or decreasing and concave down. Draw a picture to see why.)

A graph will have a vertical asymptote $x = \alpha$ if either $\lim_{x\to\alpha^+} f(x) = \infty$, $\lim_{x\to\alpha^+} f(x) = -\infty$, $\lim_{x\to\alpha^-} f(x) = \infty$ or $\lim_{x\to\alpha^-} f(x) = -\infty$. For example, if $\lim_{x\to\alpha^+} = \infty$, it follows that if a point (x, y) is on the graph and x is just a little bigger than α, then y must be very large and hence the curve must be close to the line $x = \alpha$. Similar arguments hold for each of the other cases.

Since vertical asymptotes, by their very nature, can exist only at discontinuities, it is generally a simple matter to recognize *possible* asymptotes. You can look for some of the the same clues that lead you to look for discontinuities—denominators that are zero. Once you suspect that $x = \alpha$ is a verticle asymptote, check the two one-sided limits at α.

For example, let $f(x) = x/(x-3)^2$. Clearly, the denominator is zero when $x = 3$, so $x = 3$ is a *possible* vertical asymptote. Since $\lim_{x\to3^+} x/(x - 3)^2 = \infty$, the line $x = 3$ is a vertical asymptote for the portion of the curve on the right. Since $\lim_{x\to3^-} x/(x - 3)^2 = \infty$ also, the line $x = 3$ is also vertical asymptote for the portion of the curve on the left.