Example of a Derivative That Is Not Continuous

Define $f(x) = \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$

Since $f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \dfrac{x^2 \sin(1/x) - 0}{x} = \lim_{x \to 0} x \sin(1/x) = 0$, we have $f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$

As $x \to 0$, $x \sin(1/x) \to 0$ but $\cos(1/x)$ oscillates between -1 and 1, so $f'(x)$ has no limit as $x \to 0$ and f' cannot be continuous at 0.

Proof the Derivatives Have the Intermediate Value Property

Let f be a differentiable function on an interval $[a, b]$, with $a < b$. We will show that f' takes on every value between $f'(a)$ and $f'(b)$.

Let λ be between $f'(a)$ and $f'(b)$. It suffices to show $\exists \xi \in (a, b)$ such that $f'(\xi) = \lambda$.

Define two functions α and β which are continuous on $[a, b]$ and for which $\alpha(x) < \beta(x)$ when $a < x < b$, $\alpha(x) = a$ for x close to a, $\alpha(x) = x$ for x close to b, $\beta(x) = x$ for x close to a and $\beta(x) = b$ for x close to b.

This can be done in many ways, as can be seen by drawing some pictures. For example, the graph of α *could consist of the line segments going from* (a, a) *to* $(a + \epsilon, a)$ *to* $(b - \epsilon, b - \epsilon)$ *to* (b, b), *for any positive number* $\epsilon < \dfrac{b - a}{2}$, *while the graph of* β *could go from* (a, a) *to* $(a + \epsilon, a + \epsilon)$ *to* $(b - \epsilon, b)$ *to* (b, b).

Now, define $g(x) = \dfrac{f(\beta(x)) - f(\alpha(x))}{\beta(x) - \alpha(x)}$ for $a < x < b$. Since $f(\alpha(x))$ and $f(\beta(x))$ are composites of continuous functions, g is a rational function, with non-0 denominator, of continuous functions and hence continuous on (a, b).

If x is close to a, $g(x) = \dfrac{f(x) - f(a)}{x - a}$, so $\lim_{x \to a^+} g(x) = f'(a)$. Similarly, $\lim_{x \to b^-} g(x) = f'(b)$. By the Intermediate Value Theorem for Continuous Functions, there is some x_0 between a and b such that $g(x_0) = \lambda$. Letting $A = \alpha(x_0)$ and $B = \beta(x_0)$, we have $\dfrac{f(B) - f(A)}{B - A} = \lambda$. By the Mean Value Theorem, $\exists \xi \in (A, B)$ such that $f'(\xi) = \dfrac{f(B) - f(A)}{B - A}$. Hence $f'(\xi) = \lambda$. $\square$