

Dot Products

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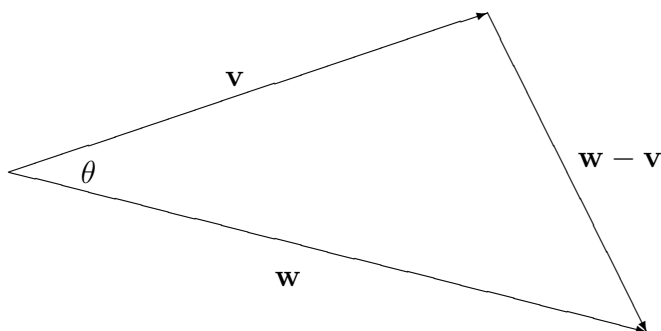
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We start with a definition for a dot product of two vectors in \mathbb{R}^n .

Definition 1 (Dot Product). *Given two vectors $\mathbf{v} = (v_1, v_2, \dots, v_n)$, $\mathbf{w} = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$, their dot product $\mathbf{v} \cdot \mathbf{w}$ is defined as $\sum_{k=1}^n v_k w_k = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$.*

Corollary 1. $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$



To see the rationale for this definition, consider two vectors $\mathbf{v} = (a, b, c)$, $\mathbf{w} = (d, e, f) \in \mathbb{R}^3$ along with their difference $\mathbf{w} - \mathbf{v} = (d - a, e - b, f - c)$ and let θ be the angle between \mathbf{v} and \mathbf{w} . If we form a triangle by placing the initial points for \mathbf{v} and \mathbf{w} at the origin, then $\mathbf{w} - \mathbf{v}$ forms the third side. If we apply the Law of Cosines, we obtain

$$\begin{aligned} |\mathbf{w} - \mathbf{v}|^2 &= |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2|\mathbf{v}| \cdot |\mathbf{w}| \cos \theta \\ (d - a)^2 + (e - b)^2 + (f - c)^2 &= a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - 2|\mathbf{v}| \cdot |\mathbf{w}| \cos \theta \\ d^2 - 2ad + a^2 + e^2 - 2be + b^2 + f^2 - 2cf + c^2 &= a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - 2|\mathbf{v}| \cdot |\mathbf{w}| \cos \theta \\ -2ad - 2be - 2cf &= -2|\mathbf{v}| \cdot |\mathbf{w}| \cos \theta \\ ad + be + cf &= |\mathbf{v}| \cdot |\mathbf{w}| \cos \theta. \end{aligned}$$

In other words, the dot product of two vectors is equal to the product of their magnitudes times the cosine of the angle between them. It is easily seen that the same result holds in \mathbb{R}^2 . In \mathbb{R}^2 and \mathbb{R}^3 , this leads immediately to the famous Cauchy-Schwarz Inequality.

Theorem 2 (Cauchy-Schwarz Inequality). $|\mathbf{v} \cdot \mathbf{w}| \leq |\mathbf{v}| \cdot |\mathbf{w}|$

The Cauchy-Schwarz Inequality can also be verified easily in \mathbb{R}^2 using ordinary algebra. It may be proven in general through the use of some calculus along with the algebraic properties of the dot product.

It may be shown via routine calculations that the dot product is commutative, distributes over vector addition and that for any scalar k and vectors \mathbf{v}, \mathbf{w} , $k(\mathbf{v} \cdot \mathbf{w}) = (k\mathbf{v}) \cdot \mathbf{w}$.

Now consider $\mathbf{v} - t\mathbf{w}$ for some scalar t . We obtain the following routine calculations.

$$\begin{aligned}
(\mathbf{v} - t\mathbf{w}) \cdot (\mathbf{v} - t\mathbf{w}) &\geq 0 \\
(\mathbf{v} - t\mathbf{w}) \cdot \mathbf{v} - (\mathbf{v} - t\mathbf{w}) \cdot t\mathbf{w} &\geq 0 \\
\mathbf{v} \cdot \mathbf{v} - 2t\mathbf{v} \cdot \mathbf{w} + t^2\mathbf{w} \cdot \mathbf{w} &\geq 0 \\
(\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w}) - 2t(\mathbf{v} \cdot \mathbf{w})(\mathbf{w} \cdot \mathbf{w}) + t^2(\mathbf{w} \cdot \mathbf{w})^2 &\geq 0
\end{aligned}$$

Since the Cauchy-Schwarz Inequality is obviously true of $\mathbf{w} = \mathbf{0}$, assume $\mathbf{w} \neq \mathbf{0}$ and let $t = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}$. Plugging that into the last inequality yields the following calculations.

$$\begin{aligned}
(\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w}) - (\mathbf{v} \cdot \mathbf{w})^2 &\geq 0 \\
(\mathbf{v} \cdot \mathbf{w})^2 &\leq (\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w}) \\
(\mathbf{v} \cdot \mathbf{w})^2 &\leq |\mathbf{v}|^2 |\mathbf{w}|^2 \\
|\mathbf{v} \cdot \mathbf{w}| &\leq |\mathbf{v}| \cdot |\mathbf{w}|.
\end{aligned}$$

This verifies the Cauchy-Schwarz Inequality and gives us a way of defining the angle between vectors in arbitrary dimensions.

Definition 2. The angle θ between vectors \mathbf{v}, \mathbf{w} is given by $\theta = \arccos \left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| \cdot |\mathbf{w}|} \right)$.

Corollary 3. $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \cdot |\mathbf{w}| \cos \theta$, where θ is the angle between \mathbf{v} and \mathbf{w} .