

Calculating Derivatives

©Alan H. Stein

The University of Connecticut at Waterbury

stein@math.uconn.edu <http://www.math.uconn.edu/~stein>

The following is a short summary of the key rules for calculating derivatives. Keep in mind that knowing how to calculate derivatives is useless unless you know what they are and how to use them.

There are two types of formulas for calculating derivatives, which we may classify as (a) formulas for calculating the derivatives of elementary functions and (b) structural type formulas. In this course we have only one of the former; students taking more calculus courses will eventually run into a number of additional formulas of this type.

FORMULAS FOR DERIVATIVES OF ELEMENTARY FUNCTIONS

$$(1) \frac{d}{dt}(t^n) = nt^{n-1}$$

Formula (1) is sometimes referred to as the "Power Rule". Note that these formulas may be used *only* to calculate derivatives of functions in precisely the *form* indicated. The name of the independent variable is irrelevant; it is the *form* that is important. For example, the power rule may be used to calculate that $\frac{d}{dx}(x^5) = 5x^4$ or that $\frac{d}{dt}(t^5) = 5t^4$, but gives us no help at all with $\frac{d}{dt}(x^5)$.

All the other formulas, the structural type formulas, reduce the task of calculating derivatives of more complicated functions into calculating several derivatives of less complicated functions. We keep using them until we finally wind up using one of the formulas for the derivatives of elementary functions.

STRUCTURAL TYPE FORMULAS

These formulas may be divided into two groups; one group is so natural that the particular formulas in it are often used without even realizing it, while the other group needs to be carefully memorized.

In the following, assume that x, y, u, v and t are variables, while c and k are constants.

The first group of formulas, which is used almost without thought, may be expressed as:

- (1) The derivative of a constant times a function equals the constant times the derivative of the function.
- (2) The derivative of a sum equals the sum of the derivatives.
- (3) The derivative of a difference equals the difference of the derivatives.

Symbolically, we write these rules as:

$$(1) \frac{d}{dt}(cu) = c \frac{du}{dt}$$
$$(2) \frac{d}{dt}(u + v) = \frac{du}{dt} + \frac{dv}{dt}$$

$$(3) \frac{d}{dt}(u - v) = \frac{du}{dt} - \frac{dv}{dt}.$$

When we apply these rules, we say that we are differentiating "term by term". Using these rules along with the power rule, it is very easy to differentiate any polynomial. Some special cases such as the following come up so often that we tend to take them for granted and use them as nonchalantly as we use the power rule:

$$(1) \frac{dc}{dt} = 0$$

$$(2) \frac{d}{dt}(ct) = c$$

$$(3) \frac{d}{dt}(at + b) = a$$

The last three rules are somewhat more difficult. They are called the product rule, the quotient rule and the chain rule. Of these, the product and quotient rules can be used routinely, since it is easy to recognize when you have a product or quotient, but it is more difficult and takes more practice to use the chain rule correctly.

The product rule may be thought of as *the derivative of a product equals the first factor times the derivative of the second plus the second factor times the derivative of the first*.

The quotient rule may be thought of as *the derivative of a quotient equals the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator*.

Symbolically, we express these rules as follows:

Formula 1 (Product Rule). $\frac{d}{dt}(uv) = u \frac{dv}{dt} + v \frac{du}{dt}$

Formula 2 (Quotient Rule). $\frac{d}{dt}(u/v) = \frac{v \frac{du}{dt} - u \frac{dv}{dt}}{v^2}$

However, we should remember the rules *in words* rather than trying to memorize jumbles of symbols.

The chain rule, as mentioned before, is a little trickier to use. Fortunately, its formula is easier to remember than some of the others.

Formula 3 (Chain Rule). $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

As usual, remember the meaning, especially since the symbols x , y and u are not necessarily the ones that will be used in the function you will be differentiating.

The chain rule is used for calculating the derivatives of composite functions. The easiest way to recognize that you are dealing with a composite function is by the process of elimination: if none of the other rules apply, then you have a composite function. *This makes it very important to be totally familiar with the other rules for differentiation.* The difficult part of using the chain rule is recognizing that you need it and then doing the algebra necessary to *decompose* the function. We have looked at two ways of doing that in class. Study them and, if you have difficulty, see your instructor. Once you have *decomposed* the function, so that you have written down something in the form

$$y = f(u)$$

$$u = g(x)$$

it is easy to actually carry out the rest of the chain rule. For example, given the function $y = \sqrt{x^2 + 5}$ you would write something like

$$y = \sqrt{u} = u^{1/2}$$

$$u = x^2 + 5$$

and then, since $\frac{dy}{du} = \frac{1}{2}u^{-1/2}$ and $\frac{du}{dx} = 2x$, we get

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2}u^{-1/2} \cdot (2x) = x/u^{1/2} = x/\sqrt{u} = x/\sqrt{x^2 + 5}.$$

SUMMARY AND OVERALL STRATEGY

The following is a general approach to calculating derivatives which, if followed, will enable you to routinely calculate every derivative we run across in the class. It depends on your being able to recognize individual terms, constant factors, products and quotients and also being able to rewrite functions in a more useful style (e.g. being able to recognize that something like $x/50$ can be rewritten as $\frac{1}{50}x$ or that \sqrt{x} can be rewritten as $x^{1/2}$).

- (1) Differentiate term by term. Deal with each term separately and, for each term, recognize any constant factor.
- (2) For each term, recognize whether it is one of the elementary functions (power, trigonometric, exponential or natural logarithm of the independent variable). If it is, you can easily apply the appropriate formula and you will be done. If it's not, go on to (3).
- (3) Decide whether the term is a product or a quotient. If it is, use the appropriate formula. *Note that the appropriate formula will have you calculating two other derivatives and you will have to go back to (1) to deal with those.* If it isn't, go to (4).
- (4) If you've gotten this far, you have to use the Chain Rule. Try to use it correctly. *The note attached to (3) applies here as well.*