Derivatives: Definition and Notation

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Let $y = f(x)$ be a function and let $c \in \mathcal{D}_f$. The following are all equivalent ways of defining *the derivative of y with respect to x at $x = c$.*

Definition 1. $f'(c) = \lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x}$

Definition 2. $f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$

Definition 3. $f'(c) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

- The name of the function need not always be f, nor must the name of the independent variable be x, nor the name of the dependent variable y. When the names are different, the definitions must be adjusted accordingly.
- The above limit, and thus the derivative, may not exist at all points in the domain of the function. If the derivative of f exists at a point c, then f is said to be *differentiable* at c.
- The process of calculating a derivative is called *differentiation*.
- Depending on the context, we may denote the derivative by $f'(c)$ or y' or $\dfrac{dy}{dx}$ as well as some other obvious variations.

Make sure that you distinguish between *the value of the derivative of a function at a point* and *the derivative function*. The word derivative is used for both, but there is a distinction.

Recall that a function is (loosely speaking) a correspondence that associates with every element of a certain set (its domain) a specific element of a second set (its codomain). When we take a function f and associate with every point $c \in \mathcal{D}_f$ at which f is differentiable the number $f'(c)$, we have in fact defined a function, which is usually denoted f' and is also called the derivative of f. Often, there is a formula that we can derive for this function.

The following example shows why it is necessary to remember the distinction.

Suppose we have the function $f(x) = x^2$ and want to find the slope of the tangent to its graph at the point $(3, 9)$. Most people remember something like the assertion *the derivative gives the slope of the tangent line.*

Question: In the above assertion, does the word *derivative* refer to the derivative *function* or the *value* of the derivative at a specific point?

Answer: It is the value of the derivative at the point of tangency. We thus solve the above example as follows.

We use the power rule for differentiation to determine the formula $f'(x) = 2x$ for the derivative *function*. We then evaluate that function at the point of tangency by calculating $f'(3) = 2 \cdot 3 = 6$ to conclude that the slope of the tangent line is 6.

Note that since we can use the formula for f to determine the second coordinate of the point of tangency, $f(3) = 3^2 = 9$, we can easily use the point-slope formula to obtain an equation for the tangent line: $y - 9 = 6(x - 3)$.