Cross Products

©Alan H. Stein
The University of Connecticut at Waterbury
stein@math.uconn.edu http://www.math.uconn.edu/∼stein

We will define another type of vector product for vectors in $\mathbb{R}^3$, to be called the cross product, which will have the following three properties.

(1) $\mathbf{v} \times \mathbf{w}$ is orthogonal (perpendicular) to both $\mathbf{v}$ and $\mathbf{w}$.
(2) $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}| \cdot |\mathbf{w}| \sin \theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.
(3) $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$ form a right-hand triple in the sense that if one turned an ordinary (right-hand) screw from the direction of $\mathbf{v}$ in the direction of $\mathbf{w}$, the screw would move forward in the direction of $\mathbf{v} \times \mathbf{w}$.

The cross product has a number of applications in the physical sciences as well as in mathematics. One immediate consequence of the third property will be that $|\mathbf{v} \times \mathbf{w}|$ is equal to the area of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$.

In order for the three properties to hold, it is necessary that the cross products of pairs of standard basis vectors are given as follows.

$$\begin{array}{ccc}
\mathbf{i} \times \mathbf{i} = \mathbf{0} & \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\
\mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{j} \times \mathbf{j} = \mathbf{0} & \mathbf{j} \times \mathbf{k} = \mathbf{i} \\
\mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{k} \times \mathbf{k} = -\mathbf{0}
\end{array}$$

Now, suppose we require the cross product to be distributive over addition and also satisfy $(k\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (k\mathbf{w}) = k(\mathbf{v} \times \mathbf{w})$ for any scalar $k \in \mathbb{R}$. We could then do the following routine, albeit tedious, calculation.

$$\begin{aligned}
(a, b, c) \times (d, e, f) &= (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) \\
&= a\mathbf{i} \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) + b\mathbf{j} \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) + c\mathbf{k} \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) \\
&= ad\mathbf{i} \times \mathbf{i} + ae\mathbf{i} \times \mathbf{j} + af\mathbf{i} \times \mathbf{k} + bd\mathbf{j} \times \mathbf{i} + be\mathbf{j} \times \mathbf{j} \\
&\quad + bf\mathbf{j} \times \mathbf{k} + cd\mathbf{k} \times \mathbf{i} + ce\mathbf{k} \times \mathbf{j} + cf\mathbf{k} \times \mathbf{k} \\
&= ad\mathbf{0} + ae\mathbf{k} - af\mathbf{j} - bd\mathbf{k} + be\mathbf{0} + bf\mathbf{i} + cd\mathbf{j} - ce\mathbf{i} + cf\mathbf{0} \\
&= (bf - ce)\mathbf{i} + (cd - af)\mathbf{j} + (ae - bd)\mathbf{k} \\
(a, b, c) \times (d, e, f) &= (bf - ce, cd - af, ae - bd)
\end{aligned}$$

This leads to our formal definition of a cross product.

Definition 1 (Cross Product)**.** $(a, b, c) \times (d, e, f) = (bf - ce, cd - af, ae - bd)$

As a mneumonic device, one can look at the cross product as the *symbolic* determinant

$$
\begin{aligned}
\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ & \mathbf{v} & \\ & \mathbf{w} & \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} b & c \\ e & f \end{vmatrix} - \mathbf{j}\begin{vmatrix} a & c \\ d & f \end{vmatrix} + \mathbf{k}\begin{vmatrix} a & b \\ d & e \end{vmatrix} \\
&= \mathbf{i}(bf - ce) - \mathbf{j}(af - cd) + \mathbf{k}(ae - bd) \\
\mathbf{v} \times \mathbf{w} &= (bf - ce, cd - af, ae - bd)
\end{aligned}
$$

It remains to be shown that this formula actually gives a vector with the properties desired.

We may start with the algebraic properties. The calculations are purely routine, albeit tedious as usual. You should be able to duplicate them yourself. We will demonstrate some of the calculations.

Claim 1. *Cross multiplication is distributive over addition*

Proof. There are really two claims here.

(1) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
(2) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

We will prove the second. The first may be demonstrated in a similar manner.

Let $\mathbf{u} = (a, b, c)$, $\mathbf{v} = (d, e, f)$, $\mathbf{w} = (g, h, i)$. Then $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (a + d, b + e, c + f) \times (g, h, i) = ((b + e)i - (c + f)h, (c + f)g - (a + d)i, (a + d)h - (b + e)g) = (bi + ei - ch - fh, cg + fg - ai - di, ah + dh - bg - eg)$.

Meanwhile, $\mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w} = (a, b, c) \times (g, h, i) + (d, e, f) \times (g, h, i) = (bi - ch, cg - ai, ah - bg) + (ei - fh, fg - di, dh - eg) = (bi + ei - ch - fh, cg + fg - ai - di, ah + dh - bg - eg)$.

A comparison of the two calculations confirms that $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$. □

Claim 2. $(k\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (k\mathbf{w}) = k(\mathbf{v} \times \mathbf{w})$ *for any scalar* $k \in \mathbb{R}$.

Proof. Let $\mathbf{v} = (a, b, c)$, $\mathbf{w} = (d, e, f)$. Then $(k\mathbf{v}) \times \mathbf{w} = (ka, kb, kc) \times (d, e, f) = (kbf - kce, kcd - kaf, kae - kbd) = k(bf - ce, cd - af, ae - bd) = k(\mathbf{v} \times \mathbf{w})$. The other part may be demonstrated in a similar manner. □

Before proving that the cross product actually has the three key properties desired, it's useful to introduce the concept of a triple product, $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$. Even without parentheses, this is unambiguous. It may be shown, with a simple but tedious calculation that we will omit, that $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$.

We first note the following method of calculating the triple product through the use of determinants.

Lemma 1. *Let* $\mathbf{u} = (a, b, c)$, $\mathbf{v} = (d, e, f)$, $\mathbf{w} = (g, h, i)$. *Then* $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$.

Proof. This may be proven with a routine calculation. $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = \mathbf{u}\cdot\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ d & e & f\\ g & h & i\end{vmatrix} = (a,b,c)\cdot$ $\left(\begin{vmatrix}e & f\\ h & i\end{vmatrix}, -\begin{vmatrix}d & f\\ g & i\end{vmatrix}, \begin{vmatrix}d & e\\ g & h\end{vmatrix}\right) = a\begin{vmatrix}e & f\\ h & i\end{vmatrix} - b\begin{vmatrix}d & f\\ g & i\end{vmatrix} + c\begin{vmatrix}d & e\\ g & h\end{vmatrix} = \begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix}$. ☐

Claim 3. $\mathbf{v}\times\mathbf{w}$ *is orthogonal (perpendicular) to both* $\mathbf{v}$ *and* $\mathbf{w}$.

Proof. Using the lemma, it follows that $\mathbf{v}\cdot\mathbf{v}\times\mathbf{w} = \begin{vmatrix}\mathbf{v}\\ \mathbf{v}\\ \mathbf{w}\end{vmatrix} = 0$ since two rows of the matrix are identical. Thus $\mathbf{v}\times\mathbf{w}$ is orthogonal to $\mathbf{v}$.

It may be shown that $\mathbf{v}\times\mathbf{w}$ is orthogonal to $\mathbf{w}$ similarly. ☐

The following identity is both interesting and useful, so we will state it as a lemma.

Lemma 2. $|\mathbf{v}\times\mathbf{w}|^2 + (\mathbf{v}\cdot\mathbf{w})^2 = |\mathbf{v}|^2|\mathbf{w}|^2$

Proof. Once again, let $\mathbf{v} = (a,b,c)$ and $\mathbf{w} = (d,e,f)$. We will begin by calculating $|\mathbf{v}\times\mathbf{w}|^2$ and $(\mathbf{v}\cdot\mathbf{w})^2$.

$|\mathbf{v}\times\mathbf{w}|^2 = |(bf-ce, cd-af, ae-bd)|^2 = (bf-ce)^2 + (cd-af)^2 + (ae-bd)^2 = b^2f^2 - 2bcef + c^2e^2 + c^2d^2 - 2acdf + a^2f^2 + a^2e^2 - 2abde + b^2d^2$.

$(\mathbf{v}\cdot\mathbf{w})^2 = (ad+be+cf)^2 = a^2d^2 + b^2e^2 + c^2f^2 + 2abde + 2acdf + 2bcef$.

If we add $|\mathbf{v}\times\mathbf{w}|^2 + (\mathbf{v}\cdot\mathbf{w})^2$, we note that all the terms with a coefficient of 2 cancel and we obtain $|\mathbf{v}\times\mathbf{w}|^2 + (\mathbf{v}\cdot\mathbf{w})^2 = a^2d^2 + a^2e^2 + a^2f^2 + b^2d^2 + b^2e^2 + b^2 + b^2f^2 + +c^2d^2 + c^2e^2 + c^2f^2 = (a^2+b^2+c^2)(d^2+e^2+f^2) = |\mathbf{v}|^2|\mathbf{w}|^2$. ☐

We now use this lemma to prove the cross product has the second property we want it to have.

Claim 4. $|\mathbf{v}\times\mathbf{w}| = |\mathbf{v}|\cdot|\mathbf{w}|\sin\theta$, *where* θ *is the angle between* $\mathbf{v}$ *and* $\mathbf{w}$.

Proof. Using the lemma, we have $|\mathbf{v}\times\mathbf{w}|^2 + (\mathbf{v}\cdot\mathbf{w})^2 = |\mathbf{v}|^2|\mathbf{w}|^2$. Since $\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}|\cdot|\mathbf{w}|\cos\theta$, it follows that $|\mathbf{v}\times\mathbf{w}|^2 + |\mathbf{v}|^2\cdot|\mathbf{w}|^2\cos^2\theta = |\mathbf{v}|^2|\mathbf{w}|^2$, so that $|\mathbf{v}\times\mathbf{w}|^2 = |\mathbf{v}|^2|\mathbf{w}|^2 - |\mathbf{v}|^2\cdot|\mathbf{w}|^2\cos^2\theta = |\mathbf{v}|^2|\mathbf{w}|^2(1-\cos^2\theta) = |\mathbf{v}|^2|\mathbf{w}|^2\sin^2\theta$. Taking square roots of both sides completes the proof. ☐

Rather than actually formally proving that $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v}\times\mathbf{w}$ form a right-hand triple, we will give a convincing intuitive argument.

Note first that $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ obviously form a right-hand triple. Since $\mathbf{k} = \mathbf{i}\times\mathbf{j}$, the triple $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{i}\times\mathbf{j}$ forms a right-hand triple.

Now imagine slowly transforming $\mathbf{i}$ into $\mathbf{v}$, with $\mathbf{i}\times\mathbf{j}$ slowly transforming into $\mathbf{v}\times\mathbf{j}$. We can visualize the triple $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ slowly transforming into the triple $\mathbf{v}$, $\mathbf{j}$, $\mathbf{v}\times\mathbf{j}$. The cross product will always be orthogonal to the other two vectors and must remain a right-hand triple since the cross product would have to reverse direction in order to have the triple transform into a left-hand triple.

We may then slowly transform $\mathbf{j}$ into $\mathbf{w}$ and $\mathbf{v}\times\mathbf{j}$ into $\mathbf{v}\times\mathbf{w}$. The triple must still remain a right-hand triple.

It is interesting to imagine how one would rigorously define a right-hand triple. In typical mathematical fashion, we recognize some algebraic property that seems to characterize it and use that property to define the concept.

Definition 2 (Right-Hand Triple)**.** *Three vectors* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{w}$ *form a right-hand triple if* $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} > 0$*.*

The sort of heuristic argument used above for the triple $\mathbf{u}$, $\mathbf{v}$, $\mathbf{v} \times \mathbf{w}$ should be fairly convincing that this really works. It also immediately follows that $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v} \times \mathbf{w}$ forms a right-hand triple since $\mathbf{v} \cdot \mathbf{w} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w}) = |\mathbf{v} \times \mathbf{w}|^2 > 0$.

Note: In most of these arguments, we have assumed that all vectors are non-$\mathbf{0}$*. Each of the claims and theorems is obviously true if any of the vectors are* $\mathbf{0}$*.*