The Quadratic Formula

The Quadratic Formula states that the solutions of a quadratic equation in the form $ax^2 + bx + c = 0$ are given by the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

This means that if $b^2 - 4ac > 0$, then there are two real solutions, $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$, if $b^2 - 4ac = 0$ there is one solution, $-\dfrac{b}{2a}$, sometimes referred to as a double root, and if $b^2 - 4ac < 0$ then there are two complex solutions, $\dfrac{-b + i\sqrt{4ac - b^2}}{2a}$ and $\dfrac{-b - i\sqrt{4ac - b^2}}{2a}$, where i represents $\sqrt{-1}$, the square root of -1.

The fact that these are solutions may be verified by plugging them back into the original equation, but that does not explain how the Quadratic Formula is obtained in the first place. One method of deriving the Quadratic Formula is through the use of the algebraic technique called *Completing the Square*. A derivation may be done as follows.

Consider the equation

$$ax^2 + bx + c = 0, \tag{1}$$

where x is a variable and a, b, c represent real numbers with $a \neq 0$. (Obviously, if $a = 0$, then the equation would not be a quadratic one.)

Each of the following equations is clearly equivalent to (1).

$$x^2 + \frac{b}{a} \cdot x + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{2}$$

Clearly, x is a solution of (2), and hence of (1), if and only if either $x + \dfrac{b}{2a} = \dfrac{b^2 - 4ac}{2a}$ or $x + \dfrac{b}{2a} = -\dfrac{b^2 - 4ac}{2a}$.

This will occur if either $x = -\dfrac{b}{2a} + \dfrac{b^2 - 4ac}{2a}$ or $x = -\dfrac{b}{2a} - \dfrac{b^2 - 4ac}{2a}$. In other words, x is a solution of the original equation if and only if either $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.