

Polynomial Factorization

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The solving of both equations and inequalities involving polynomials usually involves factoring the polynomials. In the final analysis, it doesn't matter what method is used to factor, as long as the result is correct. Although, as a practical matter, not all polynomials can be factored, the methods described below will work for virtually all polynomials we run across which can be factored.

Our method depends on the following facts from algebra:

1. A polynomial $p(x)$ has a factor $x - z$ if and only if z is a *zero* of $p(x)$, that is, if and only if $p(z) = 0$.
2. If an integer z is a zero of a polynomial $p(x)$ with integer coefficients, then z must be a divisor of the constant term of $p(x)$.
- 2a. If a rational number (remember that a rational number is a quotient of integers) z is a zero of a polynomial $p(x)$ with integer coefficients, then the numerator of z must divide the constant term of $p(x)$ while the denominator must divide the coefficient of the highest degree term of $p(x)$.

This leads to the following method for factoring a polynomial $p(x)$ with integer coefficients:

1. Find an integer zero z of $p(x)$ by checking, using trial and error, all divisors of the constant term.
 - 1a. If there are no integer zeroes, look for a rational zero by checking, using trial and error, all rational numbers where the numerator divides the constant term and the denominator divides the coefficient of the highest degree term.
2. The zero you found in (1) gives you a factor $q_1(x)$ of $p(x)$. If the zero was an integer z , then your factor is $x - z$. If the zero was a rational number p/q , then your factor is $qx - p$.
3. Now that you have one factor, it should be easy to find the other. If nothing else works, you can always divide $p(x)/q_1(x)$ to get the other factor.
4. The other factor may have more factors of its own, so repeat steps 1-3 as many times as necessary.
5. Check your factorization by multiplying out. (Using an incorrect factorization because you failed to perform this routine check is an **unpardonable** sin.

Example Factor $p(x) = x^3 - 5x + 12$ as follows:

First, check the divisors of the constant term, 12. The divisors are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. Starting with the smallest, observe that $p(1) = 10, p(-1) = 14, p(2) = 10, p(-2) = 14, p(3) = 24, p(-3) = 0$. We can stop with -3, since we have found a zero.

We now know that $x - (-3) = x + 3$ is a factor, so we write $x^3 - 5x + 12 = (x + 3) \cdot q(x)$ and we need to find $q(x)$.

By long division of $(x^3 - 5x + 12)/(x + 3)$ (or some other method), we find that $q(x) = x^2 - 3x + 4$ and thus we have factored

$$x^3 - 5x + 12 = (x + 3) \cdot (x^2 - 3x + 4).$$

We now try to factor $q(x) = x^2 - 3x + 4$ the same way. The divisors of the constant term, 4, are $\pm 1, \pm 2, \pm 4$ and these are the only possible integer zeroes of $q(x)$. Since any zero of $q(x)$ must be a zero of $p(x)$ (why?), we need only check ± 4 . Neither works and we can't factor $p(x)$ any further. (Note: we could have used the quadratic formula to see that $q(x)$ was unfactorable, since $b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot 4 = -7 < 0$.)