Inequalities

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The first step is to recognize that solving an inequality is equivalent to analyzing the sign of an expression since, for example, $f(x) < g(x)$ if and only if $f(x) - g(x)$ is negative. Thus, the first step in solving an inequality is to replace it with an equivalent inequality where one side is zero, so we merely have to analyze the sign of the other side.

Various theorems about functions together lead to the rule of thumb that if we take an expression, mark off on a number line all the points where the expression is either zero, undefined (usually because the denominator is zero), or something else strange happens, then those points divide the real line into intervals on which the expression doesn't change sign.

This suggests the following strategy for determining the sign of an expression.

Analyzing the Sign of an Expression.

(1) Find all the zeroes of the expression, along with all points where it's undefined (generally where the denominator is zero) and any other points which appear strange or in any way out of the ordinary.

(2) Mark off those points on a number line. Note that these points partition the real line into a set of intervals and that the expression maintains the same sign within each of those intervals.

(3) Determine the sign of the expression on each of those intervals. This may be done by either evaluating the expression at a single point in each interval or by simply examining the sign of each of the factors of the expression. The latter method is generally faster and more reliable. It does require that the expression be completely factored, but that must be done anyway to determine the intervals in the first place.

Example

Consider the expression $f(x) = \frac{5(x-3)(x+8)}{x(x-1)^2}$. *Note that this expression is completely factored. If the expression you are dealing with is not factored, you will need to factor it completely.*

By looking at the formula for f, it is clear that f has zeroes at 3 and at -8, and is undefined at 0 and at 1. So set up and mark off a number line as shown below.

Now take a look at each interval, one at a time.

(1) $(3, \infty)$
If $x > 3$, each factor is clearly positive, so f is positive on this interval. Alternatively, you may evaluate f at some point on this interval. For example, $f(4) = 15/4 > 0$, so f is positive at all points of the interval.

(2) $(1, 3)$
If $1 < x < 3$, then $x - 3$ is negative, since $x < 3$, but every other factor is positive, so f is negative on this interval. Alternatively, you may evaluate f at some point on this interval. For example, $f(2) = -25 < 0$, so f is negative at all points of the interval.

(3) $(0, 1)$
If $0 < x < 1$, then $x - 3$ is negative. Although $x - 1$ is negative, $(x-1)^2$ is positive, since it's a square of a non-zero number, and every other factor is positive, so f is negative on this interval. Alternatively, you may evaluate f at some point on this interval. For example, $f(1/2) = -850 < 0$, so f is negative at all points of the interval.

(4) $(-8, 0)$
If $-8 < x < 0$, then $x - 3$ is negative, $x + 8$ is positive (since $x > 8$), but x is negative while $(x-1)^2$ is positive, so f is positive on this interval. Alternatively, you may evaluate f at some point on this interval. For example, $f(-1) = 35 > 0$, so f is positive at all points of the interval.

(5) $(-\infty, -8)$
If $x < -8$, then $x - 3$ is negative, $x + 8$ is negative (since $x < -8$), and x is negative, but $(x-1)^2$ is positive, so f is negative on this interval. Alternatively, you may evaluate f at some point on this interval. For example, $f(-10) = -13/121 < 0$, so f is negative at all points of the interval.

We may visualize all this as follows.

If we wanted to express this result in standard mathematical notation, *as we always should want to*, we might say that f is positive on $(-8, 0) \cup (3, \infty)$, and f is negative on $(-\infty, -8) \cup (0, 1) \cup (1, 3)$.