Functions–Definition and Notation

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1. Definition

Definition 1 (Function)**.** *A* **function** $f : A \to B$ *is a correspondence that assigns to each element* $x \in A$ *a unique element* $y = f(x) \in B$.

The set A is called the domain of f and is often denoted by $\mathcal{D} = \mathcal{D}_f$.

The set B is called the co-domain of f.

The set $\{y \in B : y = f(x) \text{ for some } x \in \mathcal{D}_f\}$ is called the range of f and is denoted by $\mathcal{R} = \mathcal{R}_f$. (Note: The range and co-domain are often confused. Technically, they are different entities. As a practical matter, we can usually ignore the distinction.)

2. Functional Notation and Formulas

Most functions that we run across will be defined through formulas. (This is true even though the definition of a function says nothing about formulas.) The following formulas each define exactly the same function.

$$y = x^2 \tag{1}$$
$$f(x) = x^2 \tag{2}$$
$$f(t) = t^2 \tag{3}$$

In formulas 1 and 2, x is the independent variable, while in formula 3, t is the independent variable. In formula 1, y is the dependent variable, while in formulas 2 and 3, there is no dependent variable but the function is given the name f.

If we wanted to use both a dependent variable and a name for the function, we might define the function as follows.

$$y = f(x) = x^2 \tag{4}$$

It is crucial that the "functional" notation (as in formulas 2, 3 and 4) be used correctly. The key idea is that, when using "functional" notation, the **name** used in the defining formula for the independent variable is **irrelevant**. That is why we can say that formulas 2 and 3 say exactly the same thing, even though the name used for the independent variable differs.

Because of this, it makes sense to do things like defining $f(x) = x^2$ and then referring to $f(x+1)$, $f(x^2)$ or $f(x/2)$. (Quick quiz: what are they? See below.)

We can also refer to entities like $\frac{f(x+h) - f(x)}{h}$, sometimes called a difference quotient. (What is this? See below.)

One more question: What would the answers to the above questions be if the function f was defined by formula 3 rather than formula 2?

Solutions $f(x+1) = (x+1)^2$, $f(x^2) = (x^2)^2 = x^4$ and $f(x/2) = (x/2)^2$.

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} \text{ (of course).}$$

The answers remain the same if formula 3 is used rather than formula 2, since the name of the independent variable is irrelevant—the function is exactly the same.