The Wonderful World of Binomial Coefficients and Probability

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$\binom{n}{k}$ is also the k^{th} entry in the n^{th} row of Pascal's Triangle, where we count both rows and columns starting with 0.

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A combination is a collection of objects chosen from a set.

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For convenience, we often omit the braces and sometimes refer to the combination b, c or even just bc.

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bc and cb are different permutations, but they come from the same combination.

We often need to count the number of combinations of a certain size chosen from a set of a certain size.

Notation: $C_{n,k}$ is the number of combinations of k elements chosen from a set of size n.

There is a formula for $C_{n,k}$, which depends on counting permutations and on . . .

The Fundamental Principle of Counting

If a sequences of choices need to be made for which there are:
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there are $n_1 \cdot n_2 \cdot n_3 \cdot \ldots$ ways of making the sequence of choices.

A Deck of Cards

As one example of the Fundamental Principle of Counting, we can determine the number of cards in a standard deck. We can observe the number of cards must equal the number of ways we can choose a card. Since we can choose a card by first choosing its face value, which can be done in 13 ways, and then choosing its suit, which can be done in 4 ways, it follows that we can choose a card in $13 \cdot 4 = 52$ different ways. Thus, a deck of cards must contain 52 cards.

Counting Permutations

Let us denote the number of permutations containing k distinct elements chosen from a set of size n by $P_{n,k}$. We can find $P_{n,k}$ using the Fundamental Principle of Counting by observing:

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There are $(n-2)$ different ways of choosing the third element.
We can continue like this. When we get to the last, or k^{th} element, we have already chosen $k-1$ elements, so we have $n-(k-1) = n-k+1$ ways of choosing the last element.

Applying the Fundamental Principle of Counting, we have $n(n-1)(n-2)\ldots(n-k+2)(n-k+1)$ ways of choosing the k elements. Using *factorial notation*, we may write this as $\frac{n!}{(n-k)!}$.

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We just have the formula $P_{n,k} = \frac{n!}{(n-k)!}$. From this, we can fairly easily find $C_{n,k}$.

Counting Combinations

Using either the formula for counting permutations or simply using the Fundamental Principle of Counting, every combination of k elements can be arranged in $P_{k,k} = k!$ different ways. Thus, there must be $k!$ times as many permutations of a given size as combinations. In other words, $P_{n,k} = k!C_{n,k}$.

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Why?

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The number of ways this can be done is precisely the number of ways one can choose the k factors from which one chooses y.
By definition, this can be done in $C_{n,k}$ ways.
Thus, the coefficient of $x^{n-k}y^k$ must be $C_{n,k}$, so $C_{n,k}$ must be equal to $\binom{n}{k}$. From now on, we'll use $\binom{n}{k}$ rather than $C_{n,k}$.

Interesting Properties

If we look at Pascal's Triangle, which contains the Binomial Coefficients, the most obvious fact to most is that each term appears to be the sum of the two terms directly above it, to the left and to the right.

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Analytically, this may be expressed $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

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We can see why this is true if we visualize $(x+y)^n$ as $(x+y)^{n-1}(x+y)$ and observe the $x^{n-k}y^k$ term of $(x+y)^n$ comes from the $x^{n-k}y^{k-1}$ and $x^{n-k-1}y^k$ terms of $(x+y)^{n-1}$ when we multiply the former by y and the latter by x.

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We thus see the $x^{n-k}y^k$ term comes from
$\binom{n-1}{k-1}x^{n-k}y^{k-1} \cdot y + \binom{n-1}{k}x^{n-k-1}y^k \cdot x = \left[\binom{n-1}{k-1} + \binom{n-1}{k}\right] x^{n-k}y^k.$

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This shows that $\binom{n}{k}$, the coefficient of $x^{n-k}y^k$, is $\binom{n-1}{k-1} + \binom{n-1}{k}$.

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They are all powers of 2.
Another way of looking at it is, if we call the top row, row 0, which is reasonable because it corresponds to the coefficients of $(x + y)^0$, then the total in the n^{th} row is 2^n.

The Total in Row n is 2^n

This is easily seen because
$2^n = (1+1)^n = \binom{n}{0}1^n \cdot 1^0 + \binom{n}{1}1^{n-1} \cdot 1^1 + \binom{n}{2}1^{n-2} \cdot 1^2 + \cdots + \binom{n}{n}1^0 \cdot 1^n =$
$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

Connections to Probability

The most basic connection to probability is through the *Binomial Probability Distribution*.

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Suppose we perform the same experiment n times and the experiment satisfies the following conditions, where the performance of each experiment is referred to as a *binomial trial*, so that we may think of the complete sequence of trials as a *binomial experiment*.

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2. The result of each trial is independent of the results of the other trials.
3. There are numbers p and q with $p \geq 0$, $q \geq 0$ and $p + q = 1$ such that for each trial, the probability of the outcome S is p and the probability of outcome F is q. We may write $\Pr(P) = p$, $\Pr(F) = q$.

If we let X be the number of S's, it turns out that $\Pr(X = k) = \binom{n}{k} p^k q^{n-k}$.

Example: Tossing Coins

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In a simple example, if we toss a coin twice, the probability of getting exactly 1 head is
$\binom{2}{1}\left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^1 = \binom{2}{1}\left(\frac{1}{2}\right)^2 = \frac{2!}{1!1!} = \frac{2 \cdot 1}{1 \cdot 1} \cdot \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$.

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In other words, the chance of getting exactly one head if we toss a coin twice is $\frac{1}{2}$.

The same analysis implies that if any birth was equally likely to be a boy or a girl, then half the families with two children would have one boy and one girl.

Where does the connection come from?

If one performs the same trial n times, there are exactly $\binom{n}{k}$ ways of k trials coming out S and $n-k$ trials coming out F, with each way having a probability $p^k q^{n-k}$ of happening.

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This can be seen if we expand 0^n in the form $(1-1)^n$.

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If we look at the binomial coefficients, and look at the n^{th} row, it turns out that the number of odd binomial coefficients in that row is $2^{\alpha(n)}$, where $\alpha(n)$ is the number of 1^{s} in the binary (base 2) representation of n. (We need to look at the top row as the 0

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In fact, it turns out that $\binom{n}{k}$ is odd precisely when all the places where the binary representation of k have 1's are at places where the binary representation of n also have 1's.

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 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
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\end{array}
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The Macro Perspective

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Perhaps more surprisingly, if one took any finite set of prime numbers and started looking, row by row, at all the binomial coefficients, one would find that almost all the binomial coefficients were divisible by every prime in the set ... although it might take a few lifetimes to actually check enough rows to see it.

Binomial Coefficients Modulo 3

The odd binomial coefficients can be viewed as the binomial coefficients that are not divisible by 2. One might similarly look at the binomial coefficients not divisible by 3. We already know almost all binomial coefficients are divisible by 3, but among the ones which aren't, one might categorize those which leave a remainder of 1 when divided by 3 and those which leave a remainder of -1 when divided by 3.

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But there's more to the story.

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If we take 2 and raise it to a power equal to the number of 1's in the ternary (base 3) expansion of n, take 3 and raise it to a power equal to the number of 2's in the ternary expansion of n, and multiply those two numbers together, we get the number of binomial coefficients in the n^{th} row which are not divisible by 3.

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More surprisingly, if we just take the first of those factors, the power of two, that happens to be just what we get if we count the number of binomial coefficients in the n^{th} row which leave a remainder of 1 when divided by 3 and subtract the number which leave a remainder of -1.

Since any power of 2 must be a positive number, this implies the interesting result that every single row contains more binomial coefficients which give a remainder of 1 than which give a remainder of -1!!!

The Inevitability of Events of Probability Zero

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Consider a light bulb sold with an average life of 1000 hours.
I will show the probability that it lasts exactly 1000 hours is 0.
A similar argument would clearly hold for every other possible time.
But the light bulb must burn out at some point, so an event with probability 0 will inevitably occur.

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But that last sum contains n terms, each at least as large as $\frac{p}{2}$, so it must total at least $n \cdot \frac{p}{2}$. Since $n > \frac{2}{p}$, this must be more than 1. But no probability can be bigger than 1. It follows the assumption $Pr(1000) > 0$ must be wrong!

The Problem of the Three Chests

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Hint 2: At that time, his son was a reporter for the New York Times.

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Suppose you are standing in front of three chests which look exactly alike, each of the chests containing two drawers. We will call the chests A,B, and C. Each drawer of Chest A contains a gold piece. In Chest B one drawer has a gold piece, one a silver piece. In Chest C each drawer contains a silver piece.

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You choose a chest at random, choose one of its drawers at random, open the drawer, and find it contains a gold piece.

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You choose a chest at random, choose one of its drawers at random, open the drawer, and find it contains a gold piece.

What is the probability that, if you open the other drawer in the same chest, it will also contain a gold piece?