Mathematics 227 Solutions
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Due Monday, December 10, 2001 This problem set is worth 50 points.

Make sure that you check the course website for instructions, fill out the pledge form and hand it in with your paper. *No paper will be accepted without a signed pledge form.* Remember that your paper may be handed in before the deadline but that no late papers will be accepted regardless of the reason. The course website also includes an explanation of how your average will be calculated if you fail to complete this assignment.

(1) For each vector space, find its dimension and a basis.
  (a) The set of 3×2 matrices.
  Solution: The dimension is 6. The set of 3×2 matrices with all entries equal to 0 except for a single entry of 1 form a basis.
  (b) The set of vectors from the origin to points in the plane $2x + 3y - 4z = 0$.
  Solution: If we rewrite the equation in the equivalent form $x = 2z - \frac{3}{2}y$, essentially seeing that we can take y and z as free variables and x as a basic variable, we see every vector in the plane is of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2z - \frac{3}{2}y \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Thus the dimension is 2 and the set $\left\{ \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis.
  (c) $\{f | f : \mathbb{R} \to \mathbb{R},\ f'(x) = 3f(x)\}$.
  Solution: The general solution of the differential equation is $f(x) = ae^{3x}$. Thus, $\{e^{3x}\}$ is a basis and the dimension is 1.
  (d) The set of polynomials of degree 4 or less with no cubic term.
  Solution: $\{1, x, x^2, x^4\}$ forms a basis and the dimension is 4.
  (e) The column space of the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & -1 \\ -1 & 2 & 0 \end{bmatrix}$.
  Solution: If one reduces the matrix to echelon form, on finds that one pivots on the first two columns to obtain $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus the first two columns form a basis and the dimension is 2. *Indeed, one can see at sight that the third column is the sum of twice the first column plus the second column.*

(2) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(< a, b, c >) =< a + b, b - c >$.
  (a) Find the dimension of and a basis for the kernel of T.

Solution: One easily sees that $T(< a, b, c >) = 0$ if and only if $a = -b = -c$, so that every solution is a multiple of $< 1, -1, 1 >$. So the set $\{< 1, -1, 1 >\}$ is a basis for the kernel and the dimension is 1.
Alternatively, one may look at the matrix given in part (c) and find its null space by reducing it to echelon form.

(b) Find the dimension of and a basis for the range of T.
The range is a subset of $\mathbb{R}^2$ which contains at least two independent vectors, so consists of all of $\mathbb{R}^2$. Thus the dimension is 2 and any set of two vectors in $\mathbb{R}^2$ which are not multiples of one another will form a basis.

(c) Find the standard matrix for T.
Solution: The standard matrix for T has columns consisting of the images of the standard basis vectors. Since $T(< 1, 0, 0 >) =< 1, 0 >$, $T(< 0, 1, 0 >) =< 1, 1 >$, and $T(< 0, 0, 1 >) =< 0, -1 >$, the matrix is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

(d) Find the matrix for T if the basis for $\mathbb{R}^2$ is taken as $\{< 2, 1 >, < 1, 2 >\}$ rather than the standard basis.
Solution: Write $\mathbf{b}_1 =< 2, 1 >$, $\mathbf{b}_2 =< 1, 2 >$. We can easily see $T(< 1, 0, 0 >) =< 1, 0 >= \frac{2}{3}\mathbf{b}_1 - \frac{1}{3}\mathbf{b}_2$, $T(< 0, 1, 0 >) =< 1, 1 >= \frac{1}{3}\mathbf{b}_1 + \frac{1}{3}\mathbf{b}_2$, $T(< 0, 0, 1 >) =< 0, -1 >= \frac{1}{3}\mathbf{b}_1 - \frac{2}{3}\mathbf{b}_2$. Taking these as the columns of the matrix for T, we get $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$.

(3) Let $V = \mathcal{P}_3 = \{ax^3 + bx^2 + cx + d\}$ under the usual operations for polynomials with basis $\mathcal{B} = \{1,\ 1 + x,\ 1 + x + x^2,\ 1 + x + x^2 + x^3\}$, let $D : V \to V$ be the differentiation operator and let $T : V \to V$ be defined by $T(ax^3 + bx^2 + cx + d) = bx^3 + cx^2 + dx + a$.

(a) Verify that T is a linear operator.

(b) Find the matrix for D using the given basis.
Solution: Call the basis elements $\mathbf{b}_1 = 1$, $\mathbf{b}_2 = 1 + x$, $\mathbf{b}_3 = 1 + x + x^2$, $\mathbf{b}_4 = 1 + x + x^2 + x^4$. Then $D(\mathbf{b}_1) = 0$, $D(\mathbf{b}_2) = 1 = \mathbf{b}_1$, $D(\mathbf{b}_3) = 1 + 2x = 2\mathbf{b}_2 - \mathbf{b}_1$, $D(\mathbf{b}_4) = 3\mathbf{b}_3 - \mathbf{b}_2 - \mathbf{b}_1$. So the matrix for D is $\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(c) Find the matrix for T using the given basis.
Solution:
$T(\mathbf{b}_1) = T(1) = x = \mathbf{b}_2 - \mathbf{b}_1$
$T(\mathbf{b}_2) = T(1 + x) = x^2 + x = \mathbf{b}_3 - \mathbf{b}_1$
$T(\mathbf{b}_3) = T(1 + x + x^2) = x^3 + x^2 + x = \mathbf{b}_4 - \mathbf{b}_1$
$T(\mathbf{b}_4) = T(1 + x + x^2 + x^3) = x^3 + x^2 + x + 1 = \mathbf{b}_4$
So the matrix is $\begin{bmatrix} -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(d) Determine whether D is 1-1.

Solution: D is not 1-1 since $D(1) = 0$.

(e) Determine whether T is 1-1.

Solution: T is 1-1. One way of seeing this is by noting the determinant of its matrix is non-0. It is also easily seen that from its definition that different polynomials map into different polynomials.

(f) Determine whether D is onto.

Solution: D is clearly not onto, since x^3 is not in its range.

(g) Determine whether T is onto.

Solution: T is onto. It is also easily seen that from its definition that every polynomial is in its range.

(h) Let $\mathcal{B}_1 = \{1, x, x^2, x^3\}$. Find the change-of-coordinates matrix $\underset{\mathcal{B}_1 \leftarrow \mathcal{B}}{P}$.

The columns of the change-of-coordinates matrix are given by the coordinates of the elements of $\mathcal{B}$ in terms of the elements of $\mathcal{B}_1$. At sight, this can be seen to be $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(4) Consider the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

(a) Find $T(< 5, -2 >)$.

Solution: By multiplying $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 13 \\ -1 \end{bmatrix}$, we see $T(< 5, -2 >) =< 13, -1 >$.

(b) Solve the characteristic equation $|T - \lambda I)| = 0$, where I is the identity matrix. *The solutions are the eigenvalues for T. Denote them by λ_1 and λ_2.*

Solution: One doesn't even need Maple to solve $\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$, getting $(3-\lambda)^2 - 1 = 0$, so $\lambda^2 - 6\lambda + 8 = 0$ and $(\lambda - 4)(\lambda - 2) = 0$, so the solutions are $\lambda_1 = 2$, $\lambda_2 = 4$.

(c) Find vectors $\mathbf{v}_i$ such that $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ for $i = 1, 2$. *These vectors are the eigenvectors for T.*

Solution: For $\lambda = \lambda_1 = 2$, the equation $(T\lambda I)(\begin{bmatrix} x \\ y \end{bmatrix}) = \mathbf{0}$ becomes

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. It is easily solved, at sight, giving a general solution $y \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so we can take $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Similarly, for $\lambda = \lambda_2 = 4$, the equation $(T - \lambda I)(\begin{bmatrix} x \\ y \end{bmatrix}) = \mathbf{0}$ becomes

$\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$. It is easily solved, at sight, giving a general solution $y\begin{bmatrix}1\\ 1\end{bmatrix}$, so we can take $\mathbf{v}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$.

(d) Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and find the matrix representation for T with respect to the basis $\mathcal{B}$.

Solution: Since $T(\mathbf{v}_1) = 2\mathbf{v}_1$ and $T(\mathbf{v}_2) = 4\mathbf{v}_2$, the matrix representation is $\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}$.