Mathematics 227 Name: ____________________
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Due Wednesday, November 28, 2001 Solutions

(1) Determine whether each of the following sets forms a vector space under the usual operations of addition (+) and scalar multiplication (·). *Note that, of course, the conclusion must be backed up by an appropriate explanation. In particular, if the set forms a vector space, it must be shown that the properties of a vector space hold, while if the set does not form a vector space, it must be shown that at least one property of a vector space fails to hold.*

(a) The set of polynomials of degree 3.
Solution: This is not a vector space since it does not contain an additive identity.

(b) The set of polynomials of degree no greater than 3.
Solution: This is a vector space. It is a subset of the set of all polynomials, which is itself a vector space, and is closed under both addition and scalar multiplication.

(c) The set of polynomials of degree no greater than 3 with rational coefficients.
Solution: This is a vector space. It is a subset of the set of all polynomials, which is itself a vector space, and is closed under both addition and scalar multiplication.

(d) The set of 3×3 matrices.
Solution: This is a vector space. It is easy to verify each of the required properties.

(e) The set of 3×3 matrices with non-zero determinant.
Solution: This is not a vector space. In particular, it does not contain a $\mathbf{0}$ vector.

(f) The set of 3×3 matrices with zero determinant.
Solution: This is not a vector space. In particular, it is not closed under addition. For example, both $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ have zero determinant but their sum is I_3 which has determinant 1.

(g) The set of functions defined and differentiable on $[0, 1]$ whose derivatives are identically 0.
Solution:

(2) Prove that if $\mathbf{v}$ is an element of a vector space, then $(-1) \cdot \mathbf{v} = -\mathbf{v}$.
Solution: It suffices to show that $\mathbf{v} + (-1) \cdot \mathbf{v} = \mathbf{0}$. This can be shown by the following routine calculation: $\mathbf{v} + (-1) \cdot \mathbf{v} = 1 \cdot \mathbf{v} + (-1) \cdot \mathbf{v} = (1 + (-1)) \cdot \mathbf{v} = 0 \cdot \mathbf{v} = \mathbf{0}$. *If one wants further proof that* $0 \cdot \mathbf{v} = \mathbf{0}$, *note that* $0 \cdot \mathbf{v} = (0 + 0) \cdot \mathbf{v} = 0 \cdot \mathbf{v} + 0 \cdot \mathbf{v}$.

(3) Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly dependent set of vectors which span a vector space V, then there is a proper subset of S which also spans V.
Solution: Let $\mathbf{v} \in V$. It suffices to show that $\mathbf{v}$ can be written as a linear combination of a proper subset of the elements of S. Since S spans V, $\mathbf{v}$ is a linear combination of elements of S, so we can write $\mathbf{v} = \sum_{i=1}^{4} c_i \mathbf{v}_i$ for some scalars c_1, c_2, c_3, c_4. Since S is a linearly dependent set, at least one of its elements is a linear combination of the others. Without loss of generality, we may assume $\mathbf{v}_4$ is a linear combination of the others and write $\mathbf{v}_4 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ for some scalars a_1, a_2, a_3. Thus, $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = (c_1 + c_4a_1)\mathbf{v}_1 + (c_2 + c_4a_2)\mathbf{v}_2 + (c_3 + c_4a_3)\mathbf{v}_3$. So $\mathbf{v}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2\mathbf{v}_3\}$.

(4) Consider the matrix $A = \begin{pmatrix} 2 & -3 & 1 \\ 4 & 2 & -5 \end{pmatrix}$.
   (a) Find a set of linearly independent vectors which span $\mathcal{N}(A)$.
   (b) Find a set of linearly independent vectors which span the column space of A.

   Solution: We use Maple to solve $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

   >a:=matrix([[2, -3, 1], [4, 2, -5]]); b:=matrix([[0], [0]]); c:=augment(a,b);gaussjord(c);

$$a := \begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & -5 \end{bmatrix}$$

$$b := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c := \begin{bmatrix} 2 & -3 & 1 & 0 \\ 4 & 2 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{-13}{16} & 0 \\ 0 & 1 & \frac{-7}{8} & 0 \end{bmatrix}$$

We thus read o the solution $x_1 = \frac{13}{16}x_3$, $x_2 = \frac{7}{8}x_3$ and write the general solution as $\begin{bmatrix} \frac{13}{16}x_3 \\ \frac{7}{8}x_3 \\ x_3 \end{bmatrix} = x_3 \cdot \begin{bmatrix} \frac{13}{16} \\ \frac{7}{8} \\ 1 \end{bmatrix}$ and recognize that the null space is the set of multiples of the vector $\begin{bmatrix} \frac{13}{16} \\ \frac{7}{8} \\ 1 \end{bmatrix}$ or, equivalently, of $\begin{bmatrix} 13 \\ 14 \\ 16 \end{bmatrix}$.

The columns are linearly dependent and each is a linear combination of the other two while none is a multiple of any of the others, so any set of two of the three columns will form a linearly independent set of vectors spanning the column space.

(5) Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$.

(a) Find a set of linearly independent vectors which span $\mathcal{N}(A)$.

(b) Find a set of linearly independent vectors which span the column space of A.

Solution: Here we can see at sight that the rows are all multiples of one another and recognize that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in the null space precisely when $x_1 + 2x_2 + 3x_3 = 0$, so x_2 and x_3 can be viewed as free variables and any solution can be written in the form $\begin{bmatrix} -2x_2 - 3x_3 & x_2 & x_3 \end{bmatrix} = x_2 \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, so the set $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a set of linearly independent vectors spanning $\mathcal{N}(A)$.

Similarly, the columns are each multiples of the rst column, so $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a set of linearly independent vectors spanning the column space of A.

(6) Let $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$, $\mathbf{v}_3 = x^2$, $\mathbf{v}_4 = x^3$, $\mathbf{v}_5 = x^4$. Look at the set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ as a subset of the vector space of functions $f : \mathbb{R} \to \mathbb{R}$ and let W be the vector space spanned by $\mathcal{B}$.

(a) Describe W in relatively plain language.

Solution: W is the set of polynomials of degree 4 or less.

(b) Prove that $\mathcal{B}$ is a linearly independent set of vectors.

Solution: A linear combination of the elements of $\mathcal{B}$ is simply a fourth degree polynomial. If we write the linear combination as $a_0\mathbf{v}_1 + a_1\mathbf{v}_2 + a_2\mathbf{v}_3 + a_3\mathbf{v}_4 + a_4\mathbf{v}_5 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, we know from the properties of polynomials this can be identically equal to 0 only if all the coe cients are 0. Thus $\mathcal{B}$ must be a linearly independent set of vectors.

Note that this implies that $\mathcal{B}$ is a basis for W. Since a linear transformation is determined by its values on any basis, we define a linear transformation $T : W \to W$ by specifying $T(\mathbf{v}_1) = 0$, $T(\mathbf{v}_2) = \mathbf{v}_1$, $T(\mathbf{v}_3) = 2 \cdot \mathbf{v}_2$, $T(\mathbf{v}_4) = 3 \cdot \mathbf{v}_3$ and $T(\mathbf{v}_5) = 4\mathbf{v}_4$.

(c) Find $T(3\mathbf{v}_1 - 5\mathbf{v}_2 + 8\mathbf{v}_3)$.
Solution: $T(3\mathbf{v}_1 - 5\mathbf{v}_2 + 8\mathbf{v}_3) = 3T(\mathbf{v}_1 - 5\mathbf{v}_2 + 8\mathbf{v}_3 = 3T(\mathbf{v}_1) - 5T(\mathbf{v}_2) + 8T(\mathbf{v}_3) = 3 \cdot \mathbf{0} - 5 \cdot \mathbf{v}_1 + 8 \cdot 2 \cdot \mathbf{v}_2 = -5\mathbf{v}_1 + 16\mathbf{v}_2$.

(d) Find $T(2x^4 - 3x^2 + 7x - 4)$.
Solution: $T(2x^4 - 3x^2 + 7x - 4) = T(2\mathbf{v}_5 - 3\mathbf{v}_3 - +7\mathbf{v}_2 - 4\mathbf{v}_1) = 2T(\mathbf{v}_5) - 3T(\mathbf{v}_3) + 7T(\mathbf{v}_2) - 4T(\mathbf{v}_1) = 2 \cdot 4\mathbf{v}_4 - 3 \cdot 2\mathbf{v}_2 + 7\mathbf{v}_1 - 4 \cdot \mathbf{0} = 8\mathbf{v}_4 - 6\mathbf{v}_2 + 7\mathbf{v}_1$.

(e) Describe the common terminology used for the linear transformation T.
Solution: Di erentiation.

(f) Find the kernel of the linear transformation T.
Solution: The set of multiples of $\mathbf{v}_1$. In other words, the constant polynomials.

(g) Find the range of the linear transformation T.
Solution: The set of linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$. In other words, the set of polynomials of degree 3 or less.

(h) Determine whether T is one-to-one.
Solution: T is clearly not 1-1, since the image of the non-zero vector $\mathbf{v}_1$ is $\mathbf{0}$.

(i) Determine whether T is onto.
Solution: T is clearly not onto since $\mathbf{v}_5$ is clearly not in the range.

Now consider the matrix $A = (a_{i,j})_{5\times5}$, where the $a_{i,j}$ satisfy the requirement $T(\mathbf{v}_j) = \sum_{i=1}^{5} a_{i,j}\mathbf{v}_i$, $j = 1, \ldots, 5$.

(j) Find A.
Solution:

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(k) Find $|A|$.
Solution: Since the bottom row of A consists only of 0s, it's clear that $|A| = 0$.

(l) Find the null space $\mathcal{N}(A)$ of A.
Solution: One can almost at sight reduce the augmented matrix for the system $A\mathbf{x} = \mathbf{0}$ to reduced echelon form and obtain the following matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It's obvious that a $A\mathbf{x} = \mathbf{0}$ i (i.e. if and only if) $\mathbf{x}$ is a multiple of $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$.

(m) Find the column space of A.
Solution: One can also see this at sight, since the last four columns obviously form an independent set. So the column space of A is simply the span of the last four columns. Even more simply, one can look at it as the set of linear combinations of $\left\{ \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} \right\}$.