Mathematics 227 Name: ____________________
Professor Alan H. Stein
Due Monday, November 26, 2001 This problem set is worth 50 points.

Make sure that you check the course website for instructions and fill out the pledge form and hand it in with your paper. Remember that your paper may be handed in before the deadline but that no late papers will be accepted regardless of the reason. The course website also includes an explanation of how your average will be calculated if you fail to complete this assignment.

Note that, since many of the questions in this problem set can be done routinely using Maple or Mathematica, it will obviously be necessary to show, through your work, exactly how you came up with your solutions.

(1) Determine whether each of the following sets forms a vector space under the usual operations of addition (+) and scalar multiplication ($\cdot$). *Note that, of course, the conclusion must be backed up by an appropriate explanation. In particular, if the set forms a vector space, it must be shown that the properties of a vector space hold, while if the set does not form a vector space, it must be shown that at least one property of a vector space fails to hold.*
  (a) The set of polynomials of degree 3.
  (b) The set of polynomials of degree no greater than 3.
  (c) The set of polynomials of degree no greater than 3 with rational coefficients.
  (d) The set of $3 \times 3$ matrices.
  (e) The set of $3 \times 3$ matrices with non-zero determinant.
  (f) The set of $3 \times 3$ matrices with zero determinant.
  (g) The set of functions defined and differentiable on $[0, 1]$ whose derivatives are identically 0.
  (h) $\left\{ \mathbf{v} \in \mathbb{R}^2 : \begin{bmatrix} 2 & 7 \\ 5 & 3 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.
  (i) The set of linear combinations of the sin and cos functions.

(2) Prove that if $\mathbf{v}$ is an element of a vector space, then $(-1) \cdot \mathbf{v} = -\mathbf{v}$.

(3) Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly dependent set of vectors which span a vector space $V$, then there is a proper subset of $S$ which also spans $V$.

(4) Consider the matrix $A = \begin{pmatrix} 2 & -3 & 1 \\ 4 & 2 & -5 \end{pmatrix}$.

(a) Find a set of linearly independent vectors which span $\mathcal{N}(A)$.

(b) Find a set of linearly independent vectors which span the column space of $A$.

(5) Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$.

(a) Find a set of linearly independent vectors which span $\mathcal{N}(A)$.

(b) Find a set of linearly independent vectors which span the column space of $A$.

(6) Let $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$, $\mathbf{v}_3 = x^2$, $\mathbf{v}_4 = x^3$, $\mathbf{v}_5 = x^4$. Look at the set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ as a subset of the vector space of functions $f : \mathbb{R} \to \mathbb{R}$ and let $W$ be the vector space spanned by $\mathcal{B}$.

(a) Describe $W$ in relatively plain language.

(b) Prove that $\mathcal{B}$ is a linearly independent set of vectors.

*Note that this implies that $\mathcal{B}$ is a basis for $W$. Since a linear transformation is determined by its values on any basis, we define a linear transformation $T : W \to W$ by specifying $T(\mathbf{v}_1) = 0$, $T(\mathbf{v}_2) = \mathbf{v}_1$, $T(\mathbf{v}_3) = 2 \cdot \mathbf{v}_2$, $T(\mathbf{v}_4) = 3 \cdot \mathbf{v}_3$ and $T(\mathbf{v}_5) = 4\mathbf{v}_4$.*

(c) Find $T(3\mathbf{v}_1 - 5\mathbf{v}_2 + 8\mathbf{v}_3)$.

(d) Find $T(2x^4 - 3x^2 + 7x - 4)$.

(e) Describe the common terminology used for the linear transformation $T$.

(f) Find the kernel of the linear transformation $T$.

(g) Find the range of the linear transformation $T$.

(h) Determine whether $T$ is one-to-one.

(i) Determine whether $T$ is onto.

*Now consider the matrix $A = (a_{i,j})_{5\times5}$, where the $a_{i,j}$ satisfy the requirement $T(\mathbf{v}_j) = \sum_{i=1}^{5} a_{i,j}\mathbf{v}_i$, $j = 1, \ldots, 5$.*

(j) Find $A$.

(k) Find $|A|$.

(l) Find the null space $\mathcal{N}(A)$ of $A$.

(m) Find the column space of $A$.