This document contains an annotated Maple session which effectively answers most of the questions on the recent problem set along with solutions to most of the other questions.

Either Maple or Mathematica, which are similar programs for our purposes, can be used to do much of the routine calculations in Linear Algebra–and other areas of mathematics. You may use this file as a guide to using Maple. The usage of Mathematica is very similar, although the syntax and spellings are slightly different.

```
>   with(linalg);
```

Warning, new definition for norm

Warning, new definition for trace

$[BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol,$
$addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix,$
$charmat, charpoly, cholesky, col, coldim, colspace, colspan, companion, concat,$
$cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge,$
$dotprod, eigenvals, eigenvalues, eigenvectors, eigenvects, entermatrix, equal,$
$exponential, extend, ffgausselim, fibonacci, forwardsub, frobenius, gausselim,$
$gaussjord, geneqns, genmatrix, grad, hadamard, hermite, hessian, hilbert,$
$htranspose, ihermite, indexfunc, innerprod, intbasis, inverse, ismith, issimilar,$
$iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve, matadd, matrix,$
$minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace, orthog,$
$permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim,$
$rowspace, rowspan, rref, scalarmul, singularvals, smith, stackmatrix, submatrix,$
$subvector, sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose,$
$vandermonde, vecpotent, vectdim, vector, wronskian]$

Question 1: Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 10 \\ 3 & 10 & 7 & 24 \\ 5 & 16 & 1 & 61 \end{bmatrix}$, the vector $\mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ -2 \\ 4 \end{bmatrix}$ and the vector equation $A\mathbf{x} = \mathbf{b}$.

- Solve the equation by reducing the augmented matrix to echelon form.

- Solve the equation by reducing the augmented matrix to reduced echelon form.

- Solve the equation using the LU factorization for A.

- Solve the equation by finding A^{-1} and letting $\mathbf{x} = A^{-1}\mathbf{b}$.

- Find $|A|$, the determinant of A.

- Solve the equation using Cramer's Rule.

- Prove that the columns of A form a linearly independent set of vectors.

- Looking at the columns of A as a set of vectors, what is its span?

```
>   A:=matrix(4,4,[1,2,3,4,2,5,6,10,3,10,7,24,5,16,1,61]);b:=matrix(4,1,[
>   5,3,-2,4]);
```

$$A := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 10 \\ 3 & 10 & 7 & 24 \\ 5 & 16 & 1 & 61 \end{bmatrix}$$

$$b := \begin{bmatrix} 5 \\ 3 \\ -2 \\ 4 \end{bmatrix}$$

Create the augmented matrix:

```
>   aug:=augment(A,b);
```

$$aug := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 6 & 10 & 3 \\ 3 & 10 & 7 & 24 & -2 \\ 5 & 16 & 1 & 61 & 4 \end{bmatrix}$$

Reduce to echelon form:

```
>   A1:=pivot(aug,1,1);
```

$$A1 := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & -7 \\ 0 & 4 & -2 & 12 & -17 \\ 0 & 6 & -14 & 41 & -21 \end{bmatrix}$$

```
>   A2:=pivot(A1,2,2,3..4);
```

$$A2 := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & -7 \\ 0 & 0 & -2 & 4 & 11 \\ 0 & 0 & -14 & 29 & 21 \end{bmatrix}$$

```
>   A3:=pivot(A2,3,3,4..4);A4:=pivot(A3,4,4);A5:=pivot(A4,3,3);A6:=pivot(
>   A5,2,2);
```

$$A3 := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & -7 \\ 0 & 0 & -2 & 4 & 11 \\ 0 & 0 & 0 & 1 & -56 \end{bmatrix}$$

We've reduced the matrix to echelon form and can easily solve the system from here. I let Maple pivot some more to make the solution transparent; by hand, one would backsubstitute.

$$A4 := \begin{bmatrix} 1 & 2 & 3 & 0 & 229 \\ 0 & 1 & 0 & 0 & 105 \\ 0 & 0 & -2 & 0 & 235 \\ 0 & 0 & 0 & 1 & -56 \end{bmatrix}$$

$$A5 := \begin{bmatrix} 1 & 2 & 0 & 0 & \dfrac{1163}{2} \\ 0 & 1 & 0 & 0 & 105 \\ 0 & 0 & -2 & 0 & 235 \\ 0 & 0 & 0 & 1 & -56 \end{bmatrix}$$

$$A6 := \begin{bmatrix} 1 & 0 & 0 & 0 & \dfrac{743}{2} \\ 0 & 1 & 0 & 0 & 105 \\ 0 & 0 & -2 & 0 & 235 \\ 0 & 0 & 0 & 1 & -56 \end{bmatrix}$$

At this point, the solution is obvious. We now let Maple change the matrix to reduced echelon form. (This can be done with a single command, but it was done here pivot by pivot to show the individual steps.)

```
>   B1:=aug(A,b);
```

$$B1 := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 6 & 10 & 3 \\ 3 & 10 & 7 & 24 & -2 \\ 5 & 16 & 1 & 61 & 4 \end{bmatrix}$$

```
> B2:=pivot(B1,1,1);
```

$$B2 := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & -7 \\ 0 & 4 & -2 & 12 & -17 \\ 0 & 6 & -14 & 41 & -21 \end{bmatrix}$$

```
> B3:=pivot(B2,2,2);
```

$$B3 := \begin{bmatrix} 1 & 0 & 3 & 0 & 19 \\ 0 & 1 & 0 & 2 & -7 \\ 0 & 0 & -2 & 4 & 11 \\ 0 & 0 & -14 & 29 & 21 \end{bmatrix}$$

```
> B4:=pivot(B3,3,3);
```

$$B4 := \begin{bmatrix} 1 & 0 & 0 & 6 & \frac{71}{2} \\ 0 & 1 & 0 & 2 & -7 \\ 0 & 0 & -2 & 4 & 11 \\ 0 & 0 & 0 & 1 & -56 \end{bmatrix}$$

```
> B5:=pivot(B4,4,4);
```

$$B5 := \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{743}{2} \\ 0 & 1 & 0 & 0 & 105 \\ 0 & 0 & -2 & 0 & 235 \\ 0 & 0 & 0 & 1 & -56 \end{bmatrix}$$

To show how the reduction can be done in a single step:

```
> gausselim(aug);
```

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & -7 \\ 0 & 0 & -2 & 4 & 11 \\ 0 & 0 & 0 & 1 & -56 \end{bmatrix}$$

```
> gaussjord(aug);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{743}{2} \\ 0 & 1 & 0 & 0 & 105 \\ 0 & 0 & 1 & 0 & \frac{-235}{2} \\ 0 & 0 & 0 & 1 & -56 \end{bmatrix}$$

We let Maple get the LU factorization of A and use this to solve the system.

```
> U:=LUdecomp(A);Uinv:=inverse(U);L:=multiply(A,Uinv);
```

$$U := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Uinv := \begin{bmatrix} 1 & -2 & \dfrac{3}{2} & -6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & \dfrac{-1}{2} & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 6 & 7 & 1 \end{bmatrix}$$

```
> Linv:=inverse(L);Y:=multiply(Linv,b);
```

$$Linv := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 5 & -4 & 1 & 0 \\ -28 & 22 & -7 & 1 \end{bmatrix}$$

$$Y := \begin{bmatrix} 5 \\ -7 \\ 11 \\ -56 \end{bmatrix}$$

```
> X:=multiply(Uinv,Y);
```

$$X := \begin{bmatrix} \dfrac{743}{2} \\ 105 \\ \dfrac{-235}{2} \\ -56 \end{bmatrix}$$

Next, solve the system by letting $\mathbf{x} = A^{-1}\mathbf{b}$.

```
> Ainv:=inverse(A);X:=multiply(Ainv,b);
```

$$Ainv := \begin{bmatrix} \dfrac{361}{2} & -140 & \dfrac{87}{2} & -6 \\ 54 & -43 & 14 & -2 \\ \dfrac{-117}{2} & 46 & \dfrac{-29}{2} & 2 \\ -28 & 22 & -7 & 1 \end{bmatrix}$$

$$X := \begin{bmatrix} \dfrac{743}{2} \\ 105 \\ \dfrac{-235}{2} \\ -56 \end{bmatrix}$$

Next, we calculate the determinant of A, the determinants of the matrices obtained by replacing the various columns of A by $\mathbf{b}$ and find the solution using Cramer's Rule.

```
> Adet:=det(A);
```

$$Adet := -2$$

```
> X1:=augment(b,submatrix(A,1..4,2..4));X1det:=det(X1);x1:=X1det/Adet;
```

$$X1 := \begin{bmatrix} 5 & 2 & 3 & 4 \\ 3 & 5 & 6 & 10 \\ -2 & 10 & 7 & 24 \\ 4 & 16 & 1 & 61 \end{bmatrix}$$

$$X1det := -743$$

$$x1 := \frac{743}{2}$$

```
>  X2:=augment(submatrix(A,1..4,1..1),b,submatrix(A,1..4,3..4));X2det:=d
>  et(X2);x2:=X2det/Adet;
```

$$X2 := \begin{bmatrix} 1 & 5 & 3 & 4 \\ 2 & 3 & 6 & 10 \\ 3 & -2 & 7 & 24 \\ 5 & 4 & 1 & 61 \end{bmatrix}$$

$$X2det := -210$$

$$x2 := 105$$

```
>  X3:=augment(submatrix(A,1..4,1..2),b,submatrix(A,1..4,4..4));X3det:=d
>  et(X3);x3:=X3det/Adet;
```

$$X3 := \begin{bmatrix} 1 & 2 & 5 & 4 \\ 2 & 5 & 3 & 10 \\ 3 & 10 & -2 & 24 \\ 5 & 16 & 4 & 61 \end{bmatrix}$$

$$X3det := 235$$

$$x3 := \frac{-235}{2}$$

```
>  X4:=augment(submatrix(A,1..4,1..3),b);X4det:=det(X4);x4:=X4det/Adet;
```

$$X4 := \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 5 & 6 & 3 \\ 3 & 10 & 7 & -2 \\ 5 & 16 & 1 & 4 \end{bmatrix}$$

$$X4det := 112$$

$$x4 := -56$$

We have found an inverse for A, so by the Invertible Matrix Theorem it follows that the columns of A form an independent set. By the same theorem, the columns span $\mathbb{R}^4$.

Question 2: Let $\mathbf{u} =< 3, -1, 5 >$, $\mathbf{v} =< 1, -12, -5 >$, $\mathbf{w} =< 1, 2, 3 >$.

- Determine whether the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent or linearly dependent.

- Describe Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ geometrically.

We use Maple to show that the determinant of the matrix formed from the vectors is 0, which shows that the vectors are dependent. Each is a linear combination of the other two, so the span will be a plane containing the origin and containing all three vectors. It is determined by any two of the vectors.

```
>  U:=augment([3,-1,5],[1,-12,-5],[1,2,3]);det(U);
```

$$U := \begin{bmatrix} 3 & 1 & 1 \\ -1 & -12 & 2 \\ 5 & -5 & 3 \end{bmatrix}$$

$$0$$

Question 3: Let $\mathbf{u} =< 1, 1, 0 >$, $\mathbf{v} =< 1, 0, 1 >$, $\mathbf{w} =< 0, 1, 1 >$.

- Determine whether the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent or linearly dependent.

- Describe Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ geometrically.

We show that the determinant of the matrix formed by the three vectors is non-zero, so the vectors are linearly independent and span $\mathbb{R}^3$.

```
>   U:=augment([1,1,0],[1,0,1],[0,1,1]);det(U);
```

$$U := \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$-2$$

Question 4: Consider an arbitrary set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^4$. Prove that Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \neq \mathbb{R}^4$.

Proof: Suppose the span was $\mathbb{R}^4$. Then the set formed by those three vectors and the zero vector would also span $\mathbb{R}^4$, so that the matrix formed from those vectors would be invertible (by the Invertible Matrix Theorem) and have non-zero determinant. But since one column of that matrix would consist only of 0s, the determinant would have to be 0, a contradiction. So the span can't be $\mathbb{R}^4$. *This is not the only way to prove it.*

Question 5: Prove: If α, β are both solutions of the matrix equation $A\mathbf{x} = \mathbf{b}$, then $\alpha - \beta$ is a solution of the homogeneous vector equation $A\mathbf{x} = \mathbf{0}$.

Proof: A simple calculation will do. $A(\alpha - \beta) = A\alpha - A\beta = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

Question 6: Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Derive the formula for $|A|$ as we did in class, using the equivalence $|A| = D(a\mathbf{e}_1 + d\mathbf{e}_2 + g\mathbf{e}_3, b\mathbf{e}_1 + e\mathbf{e}_2 + h\mathbf{e}_3, c\mathbf{e}_1 + f\mathbf{e}_2 + i\mathbf{e}_3)$ and the properties of D.

This is a routine calculation that could be completed many different ways.

Question 7: Prove: If $A = LU$ is the LU factorization of a square matrix A, then $|A| = |U|$.

Proof: Since L is lower triangular, its determinant is simply the product of entries in its diagonal. Since all the entries in its diagonal equal 1, it follows that $|L| = 1$. Since the determinant of a product equals the product of the determinants, we get $|A| = |LU| = |L| \cdot |U| = 1 \cdot |U| = |U|$.

Question 8: Given a square nxn matrix $A = (a_{i,j})$, define $N(A) = \max_j(\sum_{i=1}^{n} a_{i,j})$. In other words, if one calculates the sums of the elements in each individual column of A, $N(A)$ is the largest of those sums. Prove that if all the entries of square matrices A and B are non-negative, the $N(AB) \leq N(A)N(B)$.

Proof Let $A = (a_{i,j})$, $B = (b_{i,j})$ be square matrices of the same order n with no negative entries. Let $C = (c_{i,j})_{nxn} = AB$. Then $c_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}$. Consider the sum S_j of terms in a fixed column j. We have $S_j = \sum_{i=1}^{n} c_{i,j} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{i,k}b_{k,j}$. We can rearrange the sum to get $S_j = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{i,k}b_{k,j}$. In the inside sum, each term has the factor $b_{k,j}$, which we can factor out (using the Distributive Law) to obtain $S_j = \sum_{k=1}^{n} b_{k,j}\sum_{i=1}^{n} a_{i,k}$. That latter sum is bounded by $N(A)$, so we have $S_j \leq \sum_{k=1}^{n} b_{k,j}N(A) = N(A)\sum_{k=1}^{n} b_{k,j} \leq N(A)N(B)$.

Question 9: Solve the Leontief production equation for an economy with three sectors, given that $C = \begin{bmatrix} .5 & .5 & .2 \\ .4 & .2 & .1 \\ .3 & .1 & .2 \end{bmatrix}$ is the consumption matrix and $\mathbf{d} = \begin{bmatrix} 50 \\ 30 \\ 60 \end{bmatrix}$ is the final demand. *You may use a symbolic manipulation package such as Maple or Mathematica to help answer this question.*

The solution is $x = (I - C)^{-1}d$, which we can easily get with Maple.

```
>   C:=matrix(3,3,[.5,.5,.2,.4,.2,.1,.3,.1,.2]);d:=matrix(3,1,[50,30,60])
>   ;
```

$$C := \begin{bmatrix} .5 & .5 & .2 \\ .4 & .2 & .1 \\ .3 & .1 & .2 \end{bmatrix}$$

$$d := \begin{bmatrix} 50 \\ 30 \\ 60 \end{bmatrix}$$

```
>   Ident:=matrix(3,3,[1,0,0,0,1,0,0,0,1]);IminusC:=evalm(Ident-C);Iminus
>   Cinv:=inverse(IminusC);x:=multiply(IminusCinv,d);
```

$$Ident := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$IminusC := \begin{bmatrix} .5 & -.5 & -.2 \\ -.4 & .8 & -.1 \\ -.3 & -.1 & .8 \end{bmatrix}$$

$$IminusCinv := \begin{bmatrix} 7.500000000 & 5.000000000 & 2.500000000 \\ 4.166666667 & 4.047619048 & 1.547619048 \\ 3.333333333 & 2.380952381 & 2.380952381 \end{bmatrix}$$

$$x := \begin{bmatrix} 675.0000000 \\ 422.6190477 \\ 380.9523810 \end{bmatrix}$$

Question 10: Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$.

- Calculate $|A|$ by hand, using elementary row and column operations until you reach the point where $|A| = k|B|$ for some constant k and a matrix B which is either upper or lower triangular so that its determinant can be calculated by simply multiplying together the entries of its diagonal.

- What can you say about the linear independence or linear dependence of the columns of A?

- What can you say about the invertibility of A?

A is actually a well known matrix, known as the *Vandermonde Matrix*, and can be easily generated with Maple. Maple can calculate its determinant very quickly, but row operations that could be used to evaluate it by hand have been shown.

```
>   A:=vandermonde([1,2,3,4]);
```

$$A := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

```
>   det(A);
```

$$12$$

```
>   A1:=addrow(A,1,2,-1);
```

$$A1 := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

```
>   A2:=addrow(A1,1,3,-1);
```

7

$$A2 := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

```
>  A3:=addrow(A2,1,4,-1);
```

$$A3 := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \\ 0 & 3 & 15 & 63 \end{bmatrix}$$

```
>  A4:=addrow(A3,2,3,-2);
```

$$A4 := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 3 & 15 & 63 \end{bmatrix}$$

```
>  A5:=addrow(A4,2,4,-3);
```

$$A5 := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 6 & 42 \end{bmatrix}$$

```
>  A6:=addrow(A5,3,4,-3);
```

$$A6 := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

From this, we easily see the determinant is $1 \cdot 1 \cdot 2 \cdot 6 = 12$. Since the determinant is not 0, the columns are independent and the matrix is invertible.