Mathematics 227   Name: _____

Professor Alan H. Stein

Due Wednesday, November 7, 2001   This problem set is worth 50 points.

Make sure that you check the course website for instructions and ll out the pledge form and hand it in with your paper. Remember that your paper may be handed in before the deadline but that no late papers will be accepted regardless of the reason. The course website also includes an explanation of how your average will be calculated if you fail to complete this assignment.

Note that, since many of the questions in this problem set can be done routinely using Maple or Mathematica, it will obviously be necessary to show, through your work, exactly how you came up with your solutions.

(1) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 10 \\ 3 & 10 & 7 & 24 \\ 5 & 16 & 1 & 61 \end{bmatrix}$, the vector $\mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ -2 \\ 4 \end{bmatrix}$ and the vector equation $A\mathbf{x} = \mathbf{b}$.

   (a) Solve the equation by reducing the augmented matrix to echelon form.

   (b) Solve the equation by reducing the augmented matrix to reduced echelon form.

   (c) Solve the equation using the LU factorization for A.

   (d) Solve the equation by nding A^{-1} and letting $\mathbf{x} = A^{-1}\mathbf{b}$.

   (e) Find $|A|$, the determinant of A.

   (f) Solve the equation using Cramer's Rule.

   (g) Prove that the columns of A form a linearly independent set of vectors.

   (h) Looking at the columns of A as a set of vectors, what is its span?

(2) Let $\mathbf{u} = <3, -1, 5>$, $\mathbf{v} = <1, -12, -5>$, $\mathbf{w} = <1, 2, 3>$.

   (a) Determine whether the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent or linearly dependent.

   (b) Describe Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ geometrically.

(3) Let $\mathbf{u} = <1, 1, 0>$, $\mathbf{v} = <1, 0, 1>$, $\mathbf{w} = <0, 1, 1>$.

   (a) Determine whether the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent or linearly dependent.

   (b) Describe Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ geometrically.

(4) Consider an arbitrary set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^4$. Prove that Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \neq \mathbb{R}^4$.

(5) Prove: If α, β are both solutions of the matrix equation $A\mathbf{x} = \mathbf{b}$, then $\alpha - \beta$ is a solution of the homogeneous vector equation $A\mathbf{x} = \mathbf{0}$.

(6) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Derive the formula for $|A|$ as we did in class, using the equivalence $|A| = D(a\mathbf{e}_1 + d\mathbf{e}_2 + g\mathbf{e}_3, b\mathbf{e}_1 + e\mathbf{e}_2 + h\mathbf{e}_3, c\mathbf{e}_1 + f\mathbf{e}_2 + i\mathbf{e}_3)$ and the properties of D.

(7) Prove: If $A = LU$ is the LU factorization of a square matrix A, then $|A| = |U|$.

(8) Given a square $n{\times}n$ matrix $A = (a_{i,j})$, de ne $N(A) = \max_j(\sum_{i=1}^n a_{i,j})$. In other words, if one calculates the sums of the elements in each individual column of A, $N(A)$ is the largest of those sums. Prove that if all the entries of square matrices A and B are non-negative, the $N(AB) \leq N(A)N(B)$.

(9) Solve the Leontief production equation for an economy with three sectors, given that

$$C = \begin{bmatrix} .5 & .5 & .2 \\ .4 & .2 & .1 \\ .3 & .1 & .2 \end{bmatrix} \text{ is the consumption matrix and } \mathbf{d} = \begin{bmatrix} 50 \\ 30 \\ 60 \end{bmatrix} \text{ is the nal demand.}$$

You may use a symbolic manipulation package such as Maple or Mathematica to help answer this question.

(10) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$.

   (a) Calculate $|A|$ by hand, using elementary row and column operations until you reach the point where $|A| = k|B|$ for some constant k and a matrix B which is either upper or lower triangular so that its determinant can be calculated by simply multiplying together the entries of its diagonal.

   (b) What can you say about the linear independence or linear dependence of the columns of A?

   (c) What can you say about the invertibility of A?