Mathematics 227 Solutions
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The first few questions refer to the following system of equations.

$$
\begin{aligned}
x + 2y + 3z &= 14 \\
2x - y + z &= 3 \\
2x + 3y + 4z &= 20
\end{aligned}
\tag{1}
$$

(1) Solve (1) using ordinary algebra.
Solution: This can be done many ways. The result is the solution $x = 1$, $y = 2$, $z = 3$.

(2) Write the augmented matrix for (1).
Solution: $\begin{bmatrix} 1 & 2 & 3 & 14 \\ 2 & -1 & 1 & 3 \\ 2 & 3 & 4 & 20 \end{bmatrix}$

(3) Solve (1) by reducing the augmented matrix to echelon form.
Solution: Calling the augmented matrix aaug, Maple gets the following echelon form.

```
> gausselim(aaug);
```

$$\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & -5 & -5 & -25 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

This leads, of course, to the same solution.

(4) Solve (1) by reducing the augmented matrix to reduced echelon form.
Solution: The solution is even more obvious here.

```
> gaussjord(aaug);
```

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(5) Write (1) as a matrix equation.
Solution: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 3 \\ 20 \end{bmatrix}$

(6) Solve (1) using matrix multiplication.
Solution: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 14 \\ 3 \\ 20 \end{bmatrix}$

Maple can be used to do the calculations:

```
> inverse(a);
```

$$\begin{bmatrix} \frac{-7}{5} & \frac{1}{5} & 1 \\ \frac{-6}{5} & \frac{-2}{5} & 1 \\ \frac{8}{5} & \frac{1}{5} & -1 \end{bmatrix}$$

```
> multiply(inverse(a),b);
```

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(7) Solve (1) using Cramer's Rule.
Solution: Of course, the solution is the same.

The next few question refer to the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$T(< x, y, z, w >) =< x + 2y, 2x - y + z, x + y + z + w > . \quad (2)$$

(8) Find the matrix for T.
Solution:

```
> t:=matrix([[1, 2, 0, 0], [2, -1, 1, 0], [1, 1, 1, 1]]);
```

$$t := \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(9) Determine whether T is 1-1.
Solution: Since $T : \mathbb{R}^4 \to \mathbb{R}^3$, T can't possibly be 1-1. This is also shown later when the null space for T is found.

(10) Determine whether T is onto.
Solution: One can see at sight that the last three columns of the matrix form a basis for $\mathbb{R}^3$, so that T most be onto. For example, the last column is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. When one looks at the next to last column, it's obvious that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is also in the column space. It's then obvious, looking at either the first or second column, that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is also in the column space. This is also shown later when Maple finds the column space.

(11) Find the null space for T.
Solution:

```
> nullspace(t);
```

$$\{[-2, 1, 5, -4]\}$$

The set of multiples of that vector is the null space.

(12) Find the column space for T.
Solution:

```
> colspace(t);
```

$$\{[0,\ 1,\ 0],\ [0,\ 0,\ 1],\ [1,\ 0,\ 0]\}$$

It follows that the column space is $\mathbb{R}^3$.

(13) Show that the set $\{<1,2>,<2,3>,<1,0>\}$ is linearly independent by expressing one of its elements as a linear combination of the others. *Extra Credit: Explain how one can immediately determine, without any calculations, that the set is linearly dependent.*
Solution: This can be done many ways. For example, since $\frac{3}{2}\cdot <1,2> = <\frac{3}{2},3>$, it is obvious that $<2,3> = \frac{3}{2}\cdot <1,2> + \frac{1}{2} <1,0>$.

For the next few questions, $\mathcal{B}_1$ represents the standard basis for $\mathbb{R}^3$ and $\mathcal{B}_2$ represents the basis $\{<1,0,0>,<2,1,0>,<3,2,1>\}$ for $\mathbb{R}^3$.

(14) Find $\underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P}$, the change-of-coordinates matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$.
Solution: We can again let Maple do the work, using the fact that the columns of the inverse of $\underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P}$ are the coordinate representations of the elements of $\mathcal{B}_1$ with respect to $\mathcal{B}_2$.

```
> pinv:=matrix([[1, 2, 3], [0, 1, 2], [0, 0, 1]]);
```

$$pinv := \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> p:=inverse(pinv);
```

$$p := \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

(15) Find the coordinate representation under $\mathcal{B}_2$ for the vector $<5,3,1>$.
Solution:

```
> multiply(p,matrix([[5], [3], [1]]));
```

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(16) Suppose $[T]_{\mathcal{B}_1} = \begin{bmatrix} 5 & -3 \\ 7 & 1 \end{bmatrix}$ and $\underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$. Find $[T]_{\mathcal{B}_2}$.
Solution: We use the fact that $[T]_{\mathcal{B}_2} = \underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P} [T]_{\mathcal{B}_1} \underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P^{-1}}$.

```
> t:=matrix([[5, -3], [7, 1]]);p:=matrix([[3, 2], [4,
> 3]]);pinv:=inverse(p);
```

$$t := \begin{bmatrix} 5 & -3 \\ 7 & 1 \end{bmatrix}$$

$$p := \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$

$$pinv := \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

```
> multiply(p,multiply(t,pinv));
```

$$\begin{bmatrix} 115 & -79 \\ 159 & -109 \end{bmatrix}$$

(17) Find the eigenvalues for the matrix $\begin{bmatrix} 3 & -1 \\ 7 & -5 \end{bmatrix}$.

Solution:

```
> eigenvals(matrix([[3, -1], [7, -5]]));
```

$$-4,\ 2$$

(18) Use the fact that $\lambda = 2, 5$ are eigenvalues for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ determined by $T(<1, 0>) = <-43, 120>$, $T(<0, 1>) = <-18, 50>$ to find a set of two linearly independent eigenvectors for T.

Solution: We'll do various parts of this several ways. First, we let Maple find the eigenvalues.

```
> a:=matrix([[-43, -18], [120, 50]]);
```

$$a := \begin{bmatrix} -43 & -18 \\ 120 & 50 \end{bmatrix}$$

```
> eigenvals(a);
```

$$2,\ 5$$

We find the eigenvalues again by having Maple get the characteristic equation and then solve it as an ordinary polynomial equation.

```
> d:=det(matrix([[-43-x, -18], [120, 50-x]]));
```

$$d := 10 - 7\,x + x^2$$

```
> solve(d,x);
```

$$2,\ 5$$

We get an eigenvector for $\lambda_1 = 2$ by letting Maple do Gaussian elimination.

```
> a1:=matrix([[-43-2, -18], [120, 50-2]]);
```

$$a1 := \begin{bmatrix} -45 & -18 \\ 120 & 48 \end{bmatrix}$$

```
> gausselim(a1);
```

$$\begin{bmatrix} -45 & -18 \\ 0 & 0 \end{bmatrix}$$

```
> gaussjord(a1);
```

$$\begin{bmatrix} 1 & \frac{2}{5} \\ 0 & 0 \end{bmatrix}$$

It is obvious that $< 2/5, -1 >$ is an eigenvector. More simply, we may prefer $< 2, -5 >$.

We then do the same with $\lambda_2 = 5$.

```
> a2:=matrix([[-43-5, -18], [120, 50-5]]);
```

$$a2 := \begin{bmatrix} -48 & -18 \\ 120 & 45 \end{bmatrix}$$

```
> gausselim(a2);
```

$$\begin{bmatrix} -48 & -18 \\ 0 & 0 \end{bmatrix}$$

```
> gaussjord(a2);
```

$$\begin{bmatrix} 1 & \frac{3}{8} \\ 0 & 0 \end{bmatrix}$$

We see that $< 3/8, -1 >$ is an eigenvector. We may prefer $< 3, -8 >$.

Maple will also get the eigenvalues and eigenvectors all at once.

```
> eigenvectors(a);
```

$$[2,\ 1,\ \{\left[1,\ \frac{-5}{2}\right]\}],\ [5,\ 1,\ \{\left[1,\ \frac{-8}{3}\right]\}]$$

We interpret this to mean that one eigenvalue is $\lambda = 2$, it has one eigenvector associated with it, and that eigenvector is $< 1, -5/2 >$. Similarly, the other eigenvalue is $\lambda = 5$, it has one eigenvector associated with it, and that eigenvector is $< 1, -8/3 >$.