Most of what follows is a Maple session that yields most of the solutions to the exam. The first line is a command to use the "linalg" package.

```
> with (linalg);
```

Here, we enter the coefficient matrix A, the constant matrix B and the augmented matrix C, answering question 2 and preparing for the later questions.

```
> A:= matrix([[1, 4, 1], [0, 1, 1], [1, 6, 2]]); B:=matrix([[3],
[9],
> [4]]);C:=augment(A,B);
```

$$A := \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 1 \\ 1 & 6 & 2 \end{bmatrix}$$

$$B := \begin{bmatrix} 3 \\ 9 \\ 4 \end{bmatrix}$$

$$C := \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & 1 & 1 & 9 \\ 1 & 6 & 2 & 4 \end{bmatrix}$$

We tell Maple to use Gauss-Jordan to reduce the augmented matrix to reduced echelon form.

```
> gaussjord(C);
```

$$\begin{bmatrix} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 17 \end{bmatrix}$$

From this, we can read off the solution to the system: $x = 18$, $y = -8$, $z = 17$.

We then ask Maple to get the matrix U in the LU Decomposition of A.

```
> U:=LUdecomp(A);
```

$$U := \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Once we have U, we *cheated* to get L. Since $A = LU$, it follows that $L = AU^{-1}$.

```
> L:=multiply(A,inverse(U));
```

$$L := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Next, we solved $LY = B$ by letting $Y = L^{-1}B$ and then solved $UX = Y$ by letting $X = U^{-1}Y$. This, of course, is not how one would do it by hand–otherwise, why would one get an LU factorization in the first place–but provides an easy way to check your work. Note the solution obtained this way is, of course, the same as the solution obtained by reducing the augmented matrix.

```
> Y:=multiply(inverse(L),B); X:=multiply(inverse(U),Y);
```

$$Y := \begin{bmatrix} 3 \\ 9 \\ -17 \end{bmatrix}$$

$$X := \begin{bmatrix} 18 \\ -8 \\ 17 \end{bmatrix}$$

Finally, we solve the same system by calculating $X = A^{-1}B$, again obtaining the same solution.

```
> AINV := inverse(A); X:=multiply(AINV,B);
```

$$AINV := \begin{bmatrix} 4 & 2 & -3 \\ -1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$X := \begin{bmatrix} 18 \\ -8 \\ 17 \end{bmatrix}$$

On to question 6. We use Gauss-Jordan represent $\mathbf{0}$ as a linear combination $\mathbf{v}$ and $\mathbf{w}$, finding $a\mathbf{v} + b\mathbf{w} = \mathbf{0}$ if $a + 3b = 0$. Effectively, b is a free variable. We can set $b = 1$, solve for $a = -3$ and get the linear combination $-3\mathbf{v} + \mathbf{w} = \mathbf{0}$.

Of course, this is overkill, since it is obvious at sight that $\mathbf{w} = 3\mathbf{v}$.

```
> v:=[2,5,3];w:=[6,15,9];zero:=[0,0,0];
```

$$v := [2,\ 5,\ 3]$$

$$w := [6,\ 15,\ 9]$$

$$zero := [0,\ 0,\ 0]$$

```
> Z:=augment(v,w,zero);
```

$$Z := \begin{bmatrix} 2 & 6 & 0 \\ 5 & 15 & 0 \\ 3 & 9 & 0 \end{bmatrix}$$

```
> gaussjord(Z);
```

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For question 7, we do the same thing with two other vectors. This time, when we try to represent $\mathbf{0}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$ we find only the trivial linear combination works, showing that $\{\mathbf{v}, \mathbf{w}\}$ forms a set of linearly independent vectors. This can also be easily seen at sight, since $a\mathbf{v} + b\mathbf{w} =< 3a, 6a + b >$. For this to be $< 0, 0 >$, it is obvious that a must equal 0, from which it is obvious that b must also equal 0.

```
> v:=[3,2];w:=[0,1];zero:=[0,0];Z:=augment(v,w,zero);
```

$$v := [3,\ 2]$$

$$w := [0,\ 1]$$

$$zero := [0, 0]$$

$$Z := \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

```
> gaussjord(Z);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Question 8 is a routine calculation.

```
> T:=matrix([[2, 5, -3], [8, -1, 4]]);v:=[5,1,2];multiply(T,v);
```

$$T := \begin{bmatrix} 2 & 5 & -3 \\ 8 & -1 & 4 \end{bmatrix}$$

$$v := [5, 1, 2]$$

$$[9, 47]$$

Question 9: The columns of the matrix of a transformation are simply the images of the standard basis vectors.

```
> T:=augment([1,3],[1,-1]);
```

$$T := \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

Question 10 is a simple calculation.

```
> A:=matrix([[2,5,1],[-1,0,2]]);B:=matrix([[1,-1],[2,3],[-1,1]]);multip
> ly(A,B);
```

$$A := \begin{bmatrix} 2 & 5 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

$$B := \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 14 \\ -3 & 3 \end{bmatrix}$$

Question 11: Using the hint, we note that, on the one hand, $A(BC) = AI = A$, since $BC = I$, while from the associative law $A(BC) = (AB)C = IC = C$. It immediately follows that $A = C$. Note that this calculation shows that if a square matrix has a left inverse and a right inverse then it is invertible, with the left and right inverses actually being equal and being the inverse.