Question 6.10: Find all subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$.

Solution: Each element of the group will generate a cyclic subgroup, although some of these will be identical. From these individual elements we get the following subgroups. (The order of each subgroup is given in parentheses.)

- (0): $< (0,0) >$
- (4): $< (0,1) >=< (0,3) >=< 0 > \times \mathbb{Z}_4$
- (2): $< (0,2) >=< 0 > \times < 2 >$
- (2): $< (1,0) >= \mathbb{Z}_2 \times < 0 >$
- (4): $< (1,1) >=< (1,3) >$
- (2): $< (1,2) >$

$\mathbb{Z}_2 \times \mathbb{Z}_4$ itself is a subgroup.

Any other subgroup must have order 4, since the order of any subgroup must divide 8 and:

- The subgroup containing just the identity is the only group of order 1.
- Every subgroup of order 2 must be cyclic.
- The only subgroup of order 8 must be the whole group.

For any other subgroup of order 4, every element other than the identity must be of order 2, since otherwise it would be cyclic and we've already listed all the cyclic groups. Since there are three elements of order 2: $(0,2), (1,0), (1,2)$, the only other subset that could possibly be a subgroup of order 4 must be $\{(0,0), (0,2), (1,0), (1,2)\} = \mathbb{Z}_2 \times < 2 >$. This is easily seen to be a group and completes our list.

We thus have eight subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$.