Mathematics 2110 <u>SOLUTIONS</u>
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1. Evaluate $\iint\limits_{\mathcal{D}} (x+y)e^{x^2-y^2}\,dA$, where $\mathcal{D}$ is the parallelogram bounded by the lines $x-y=0$, $x-y=2$, $x+y=0$, $x+y=3$. *Hint: Use a change of variables.*

   Solution: We let $u = x-y$, $v = x+y$, so $x = \dfrac{u+v}{2}$, $y = \dfrac{v-u}{2}$ and $\dfrac{@(x;y)}{@(u;v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$. We thus have $\iint\limits_{\mathcal{D}} (x+y)e^{x^2-y^2}\,dA = \int_0^3\int_0^2 ve^{uv} \cdot \dfrac{@(x;y)}{@(u;v)}\,du\,dv = \frac{1}{2}\int_0^3\int_0^2 ve^{uv}\,du\,dv = \frac{1}{2}\int_0^3 \left[e^{uv}\Big|_0^2\right] dv = \frac{1}{2}\int_0^3 e^{2v} - 1\,dv = \frac{1}{2}\left[\dfrac{e^{2v}}{2} - v\right]_0^3 = \frac{1}{2}[(\dfrac{e^6}{2} - 3) - (\dfrac{1}{2} - 0)] = \dfrac{e^6}{4} - \dfrac{7}{4}$.

2. Determine whether $\mathbf{F}(x;y) = (2x\sin y - \sin x)\mathbf{i} + (x^2\cos y)\mathbf{j}$ is a conservative vector eld. If it is a conservative vector eld, nd a potential function $(x;y)$ such that $\nabla = \mathbf{F}$.

   Solution: $\dfrac{@}{@y}(2x\sin y - \sin x) = 2x\cos y$ and $\dfrac{@}{@x}(x^2\cos y) = 2x\cos y$. Since these are equal, it is a conservative vector eld.

   $(x;y) = x^2\sin y + \cos x$.

3. Calculate $\int\limits_{\mathcal{C}} y\,dx + z\,dy + x\,dz$, where $\mathcal{C}$ is the line segment from the origin to the point $(1;1;1)$.

   Solution: We may parametrize $\mathcal{C}$ as $x = t; y = t; z = t$, $0 \le t \le 1$ to get $\int\limits_{\mathcal{C}} y\,dx + z\,dy + x\,dz = \int_0^1 3t\,dt = \dfrac{3t^2}{2}\Big|_0^1 = \frac{3}{2}$.

4. Calculate $\int\limits_{\mathcal{C}} y\,dx + z\,dy + x\,dz$, where $\mathcal{C}$ is the union of the line segment going from the origin to the point $(1;0;0)$ to the point $(1;1;0)$ to the point $(1;1;1)$.

   Solution: $\int\limits_{\mathcal{C}} y\,dx + z\,dy + x\,dz = \int_0^1 0\,dx + \int_0^1 0\,dy + \int_0^1 1\,dz = 0 + 0 + 1 = 1$.

5. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $y \geq 0$. If the linear density is $k|x|y$, for some constant $k > 0$, nd the mass and center of mass of the wire.

   Solution: We may parametrize the wire $\mathcal{W}$ via $x = 2\cos t$, $y = 2\sin t$, $0 \leq t \leq$.

   The mass is $\int\limits_{\mathcal{W}} k|x|y\, ds = \int_0^\pi k(2|\cos t|)(2\sin t)\sqrt{(\pm\sin t)^2 + (\cos t)^2}\, dt =$

   $4k\int_0^\pi \sin t|\cos t|\, dt = 8k\int_0^{\pi/2} \sin t\cos t\, dt = 4k\sin^2 t\big|_0^{\pi/2} = 4k(1^2 - 0^2) = 4k.$

   $M_y = \int\limits_{\mathcal{W}} x \cdot k|x|y\, ds = \int_0^\pi k(2|\cos t|)(2|\cos t|)(2\sin t)\sqrt{(\pm\sin t)^2 + (\cos t)^2}\, dt =$

   $8k\int_0^\pi \sin t\cos^2 t\, dt = \frac{8k\cos^3 t}{3}\Big|_0^\pi = 0.$

   $M_x = \int\limits_{\mathcal{W}} y \cdot k|x|y\, ds = \int_0^\pi k(2\sin t)(2|\cos t|)(2\sin t)\sqrt{(\pm\sin t)^2 + (\cos t)^2}\, dt =$

   $8k\int_0^\pi \sin^2 t|\cos t|\, dt = 16k\int_0^{\pi/2} \sin^2 t\cos t\, dt = \frac{16k\sin^3 t}{3}\Big|_0^{\pi/2} = \frac{16k}{3} - 0 = \frac{16k}{3}.$

   Hence, $\overline{x} = \frac{M_y}{M} = \frac{0}{4k} = 0$, $\overline{y} = \frac{M_x}{M} = \frac{16k=3}{4k} = \frac{4}{3}$ and the center of mass is at $(0; \frac{4}{3})$.

6. Calculate $\int\limits_{\mathcal{C}} (\ln y + 2xy^3)\, dx + (3x^2y^2 + x=y)\, dy$, where $\mathcal{C}$ is the path from $(0;1)$ to $(2;5)$ along the parabola $y = x^2 + 1$.

   Solution: The integral is independent of path, with potential function $(x;y) = x\ln y + x^2y^3$, so $\int\limits_{\mathcal{C}} (\ln y + 2xy^3)\, dx + (3x^2y^2 + x=y)\, dy = (2;5) - (0;1) =$ $2\ln 5 + 2^2 5^3 - (0\ln 1 + 0^2 1^3) = 2\ln 5 + 500.$

7. Use a line integral to show the area of a unit circle is .

   Solution: Parametrizing the unit circle $\mathcal{C}$ by $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2$, we get the area is $\int_{\mathcal{C}} x\, dy = \int\limits_0^{2\pi} (\cos t)(\cos t)\, dt = \frac{1}{2}\int\limits_0^{2\pi} \cos^2 t + \sin^2 t\, dt = \frac{1}{2}\int\limits_0^{2\pi} 1\, dt = \frac{1}{2} \cdot 2 =$.

   Of course, $\int\limits_0^{2\pi} \cos^2 t\, dt$ *could have been evaluated many other ways.*

8. Use Green's Theorem to evaluate $\int\limits_{\mathcal{C}} \sin y\, dx + x\cos y\, dy$, where $\mathcal{C}$ is the ellipse $x^2 + xy + y^2 = 1$, oriented counterclockwise.

   Solution: Letting $\mathcal{D}$ be the region bounded by the ellipse, we have $\int\limits_{\mathcal{C}} \sin y\, dx + x\cos y\, dy =$ $\int\limits_{\mathcal{D}} \frac{@}{@x}(x\cos y) - \frac{@}{@x}(\sin y)\, dA = \int\limits_{\mathcal{D}} 0\, dA = 0.$ *Of course, we could have just as easily recognized the integral was independent of path.*

9. Find a parametrization of the unit sphere with center at the origin.

   Solution: $x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos u$
   $0 \le u \le$, $0 \le v \le 2$.

10. Find a parametrization of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$.

    Solution: $x = 2 \sin u \cos v$, $y = 3 \sin u \sin v$, $z = 5 \cos u$
    $0 \le u \le$, $0 \le v \le 2$.

11. Find a parametrization of a cone of radius 5 and height 20. *Hint: It will probably be easiest if you orient the cone with its vertex at the origin and hold it the way you'd hold an ice cream cone if you didn't want the ice cream to fall out of the cone.*

    Solution: $x = u \cos v$, $y = u \sin v$, $z = 4u$
    $0 \le u \le 5$, $0 \le v \le 2$
    We simply took $z = 4r$, *using cylindrical coordinates, and then replaced* r *by* u *and by* v. *Note that these replacements were unnecessary.*

12. Find a parametrization of a cylinder of height 20 and radius 5.

    Solution: $x = 5 \cos u$, $y = 5 \sin u$, $z = v$
    $0 \le u \le 2$, $0 \le v \le 20$.

13. Evaluate the surface integral $\iint_S xyz\, dS$, where S is the triangular region with vertices $(1;0;0)$, $(0;1;0)$, $(0;0;1)$.

    Solution: We may parametrize the surface in terms of x and y, with $z = 1 - x - y$ above the triangle in the xy-plane bounded by the two coordinate axes and the line $x + y = 1$ (or $y = 1 - x$).

    We thus have $\mathbf{r} = \langle x;y;1-x-y \rangle$, $\mathbf{r}_x = \langle 1;0;-1 \rangle$, $\mathbf{r}_y = \langle 0;1;-1 \rangle$, $\mathbf{r}_x \times \mathbf{r}_y = \langle 1;1;1 \rangle$, $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{3}$.

    We thus obtain $\iint_S xyz\, dS = \int_0^1 \int_0^{1-x} xy(1-x-y)\sqrt{3}\, dy\, dx = \sqrt{3} \int_0^1 \int_0^{1-x} xy - x^2 y - xy^2\, dy\, dx = \sqrt{3} \int_0^1 \left[xy^2=2 - x^2y^2=2 - xy^3=3 \right]_0^{1-x} dx = \sqrt{3} \int_0^1 x(1-x)^2=2 - x^2(1-x)^2=2 - x(1-x)^3=3\, dx = \sqrt{3} \int_0^1 x=6 + x^3=2 - x^4=6\, dx = \sqrt{3} \left[x^2=12 + x^4=8 - x^5=30 \right]_0^1 = \sqrt{3}(1=12 + 1=8 - 1=30) = \frac{7}{40}$.

14. Use Stokes' Theorem to evaluate $\int_{\mathcal{C}} e^{-x}\,dx + e^{x}\,dy + e^{z}\,dz$, where $\mathcal{C}$ is the triangle with vertices $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 1)$, oriented counterclockwise when viewed from above.

    Solution: Letting $\mathcal{D}$ be the interior and boundary of the triangle, we may parametrize $\mathcal{D}$ by $x = v$, $y = 2 - u - v$, $z = u$, with $0 \le u \le 1$, $0 \le v \le 1 - u$.

    Write $\mathbf{r} = \langle v, 2 - u - v, u\rangle$. We have $\mathbf{r}_u = \langle 0, -1, 1\rangle$, $\mathbf{r}_v = \langle 1, -1, 0\rangle$, $\mathbf{r}_u \times \mathbf{r}_v = \langle 1, 1, 1\rangle$.

    Using Stokes' Theorem, $\int_{\mathcal{C}} e^{-x}\,dx + e^{x}\,dy + e^{z}\,dz = \iint_{\mathcal{D}} (\nabla \times \langle e^{-x}, e^{x}, e^{z}\rangle) \cdot \mathbf{N}\,dS = \iint_{\mathcal{D}} (\nabla \times \langle e^{-x}, e^{x}, e^{z}\rangle) \cdot \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \iint_{\mathcal{D}} (\nabla \times \langle e^{-x}, e^{x}, e^{z}\rangle) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv = \int_0^1 \int_0^{1-u} \langle 0, 0, e^{v}\rangle \cdot \langle 1, 1, 1\rangle\,dv\,du = \int_0^1 \int_0^{1-u} e^{v}\,dv\,du$. (We use the fact that $e^{x} = e^{v}$ since $x = v$.)

    Since $\int_0^{1-u} e^{v}\,dv = e^{v}\Big|_0^{1-u} = e^{1-u} - 1$, we have $\int_{\mathcal{C}} e^{-x}\,dx + e^{x}\,dy + e^{z}\,dz = \int_0^1 e^{1-u} - 1\,du = -e^{1-u} - u\Big|_0^1 = (-e^{0} - 1) - (-e - 0) = e - 2$.