Mathematics 2110 SOLUTIONS
Professor Alan H. Stein
Due Monday, September 22, 2008 Problem Set

1. Find vector parametric, scalar parametric and scalar symmetric equations for the line through the points $(5, 3, 1)$ and $(2, 7, 9)$.

   Solution:

   Vector Parametric: $\mathbf{x} = \langle 5, 3, 1\rangle + \langle -3, 4, 8\rangle t$

   Scalar Parametric: $x = 5 - 3t$, $y = 3 + 4t$, $z = 1 + 8t$

   Scalar Symmetric: $\frac{x-5}{-3} = \frac{y-3}{4} = \frac{z-1}{8}$

2. Find the center and radius of the sphere $x^2 + y^2 + 6y + z^2 = 4x + 8z + 200$.

   Solution: Completing the Square: $x^2 - 4x + y^2 + 6y + z^2 - 8z = 200$,
   $(x-2)^2 - 4 + (y+3)^2 - 9 + (z-4)^2 - 16 = 200$, $(x-2)^2 + (y+3)^2 + (z-4)^2 - 16 = 229$.
   So the center is $(2, -3, 4)$ and the radius is $\sqrt{229}$.

3. Find the angle between the vectors $\langle 5, 3, 9\rangle$, $\langle 4, 2, 7\rangle$.

   Solution: $\langle 5, 3, 9\rangle \cdot \langle 4, 2, 7\rangle = 5 \cdot 4 + 3 \cdot 2 + 9 \cdot 7 = 89$.
   $|\langle 5, 3, 9\rangle| = \sqrt{115}$, $|\langle 4, 2, 7\rangle| = \sqrt{69}$, so the angle is $\arccos(\frac{89}{\sqrt{115 \cdot 69}})$.

4. Find the scalar projection of $\langle 5, 3, 9\rangle$ on $\langle 4, 2, 7\rangle$.

   Solution: $\frac{\langle 5,3,9\rangle \cdot \langle 4,2,7\rangle}{|\langle 4,2,7\rangle|} = \frac{89}{\sqrt{69}}$.

5. Find the vector projection of $\langle 5, 3, 9\rangle$ on $\langle 4, 2, 7\rangle$.

   Solution: $\frac{\langle 5,3,9\rangle \cdot \langle 4,2,7\rangle}{\langle 4,2,7\rangle \cdot \langle 4,2,7\rangle}\langle 4, 2, 7\rangle = \frac{89}{69}\langle 4, 2, 7\rangle$

6. Find an equation for the plane containing the points $(0, 1, 2)$, $(1, 2, 3)$ and $(3, 2, 1)$.

   Solution: Find a normal vector $(\langle 1, 2, 3\rangle - \langle 0, 1, 2\rangle) \times (\langle 3, 2, 1\rangle - \langle 1, 2, 3\rangle) = \langle 1, 1, 1\rangle \times \langle 2, 0, -2\rangle = \langle -2, 4, -2\rangle$

   We could use that, but it will be more convenient to use $\mathbf{n} = \langle 1, -2, 1\rangle$.

   We get equation $\mathbf{n} \cdot \langle x, y, z\rangle = \mathbf{n} \cdot \langle 1, 2, 3\rangle$, or $x - 2y + z = 0$.

7. Find the distance between the point $(2, 3, 4)$ and the plane $2x + 3y + 4z = 5$.

   Solution: Given a point $\mathbf{x}_0$ on the plane, the distance will be equal to the magnitude of the scalar projection of $\langle 2, 3, 4\rangle - \mathbf{x}_0$ on the normal vector $\langle 2, 3, 4\rangle$. The scalar projection is $\frac{\mathbf{n} \cdot (\langle 2,3,4\rangle - \mathbf{x}_0)}{|\mathbf{n}|} = \frac{\mathbf{n} \cdot \langle 2,3,4\rangle - \mathbf{n} \cdot \mathbf{x}_0}{\sqrt{29}} = \frac{29-5}{\sqrt{29}} = \frac{24}{\sqrt{29}}$. So the distance is $\frac{24}{\sqrt{29}}$.

(8-15): Consider the vector function $\mathbf{x} = \langle 2t\cos t, 2t\sin t, t\rangle$.

8. Sketch the graph of $\mathbf{x}$ and describe it in relatively plain language.

   Solution: The curve travels around the cone $4z^2 = x^2 + y^2$.

9. Find $\mathbf{v}$ and $\mathbf{a}$.

   Solution:

   $\mathbf{v} = \langle 2\cos t - 2t\sin t, 2\sin t + 2t\cos t, 1\rangle$

   $\mathbf{a} = \langle -4\sin t - 2t\cos t, 4\cos t - 2t\sin t, 0\rangle$

10. Find $\mathbf{T}$.

    Solution: $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 2\cos t - 2t\sin t, 2\sin t + 2t\cos t, 1\rangle}{\sqrt{4t^2+5}}$

11. Find $\mathbf{N}$.

    Solution:

    Too messy to print out. To calculate it, find $\mathbf{T}'$ and divide by its length.

12. Find $\mathbf{B}$.

    Solution: Also too messy to print out. $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

13. Describe the osculating and normal planes at arbitrary points on the curve.

    Solution: If you're looking directly at the point, the osculating plane will be tilted and seem to contain a small portion of the curve, with the normal plane intersecting it at a right angle. *Yes, I know - not a very good description.*

14. Find the tangential and normal components of $\mathbf{a}$.

    Solution:

    The tangential component is $\mathbf{a} \cdot \mathbf{T} = \frac{4t}{\sqrt{4t^2+5}}$.

    The normal component is $\mathbf{a} \cdot \mathbf{N} = 2\sqrt{\frac{4t^4+17t^2+20}{5+4t^2}}$.

15. Find the curvature κ.

    Solution: $\kappa = 2\sqrt{\frac{4t^4+17t^2+20}{(5+4t^2)^3}}$.