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Usually the axes are called x, y and z, but that isn't essential. The three axes form a *right hand system*, in the sense that if one uses a screwdriver on a screw, turning clockwise from the x-axis towards the y-axis, the screw moves in the direction of the z-axis.

Coordinates of a Point

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Drawing the Coordinate Axes

To do the impossible and draw the three perpendicular axes in a plane, we draw the y-axis going horizontally to the right, the z-axis vertically going up, and the x-axis making an angle of $\frac{3\pi}{8}$ or 135° with the other two axes.

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Physicists and engineers sometimes draw the x and y-axes where they're drawn for $\mathbb{R}^2$ and the z-axis where we draw the x-axis.

Distance Between Two Points

Consider points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$.

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Equations of Spheres

Since a sphere with center (x_0, y_0, z_0) and radius r consists of all points (x, y, z) a distance r from (x_0, y_0, z_0), the distance formula immediately shows its equation is

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Example: $(x + 4)^2 + (y - 2)^2 + (z - 8)^2 = 43$ is an equation for the sphere with center $(-4, 2, 8)$ and radius $\sqrt{43}$.

Completing the Square

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Thus, $x^2 + 6x + y^2 - 8y + z^2 + 14z = 7$ may be written in the form $[(x + 3)^2 - 9] + [(y - 4)^2 - 16] + [(z + 7)^2 - 49] = 7$, or

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Thus, $x^2 + 6x + y^2 - 8y + z^2 + 14z = 7$ may be written in the form $[(x + 3)^2 - 9] + [(y - 4)^2 - 16] + [(z + 7)^2 - 49] = 7$, or $(x + 3)^2 + (y - 4)^2 + (z + 7)^2 = 81$.

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Thus, $x^2 + 6x + y^2 - 8y + z^2 + 14z = 7$ may be written in the form $[(x + 3)^2 - 9] + [(y - 4)^2 - 16] + [(z + 7)^2 - 49] = 7$, or $(x + 3)^2 + (y - 4)^2 + (z + 7)^2 = 81$.

So the equation is for a sphere with center $(-3, 4, -7)$ and radius 9.

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Most of our early examples will be in $\mathbb{R}^2$, but will easily generalize to $\mathbb{R}^3$ or higher dimensional Euclidean space.

Geometric Interpretation and Notation

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Notation: Vectors are usually printed in boldface, such as $\mathbf{v} =< a, b >$. It's hard to print in boldface, so when writing vectors by hand one generally puts an arrow above it, such as $\overrightarrow{v} =< a, b >$.

Addition of Vectors

Definition (Vector Addition)

$< a, b > + < c, d > = < a + c, b + d >$.

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This probably isn't much of a surprise. *This definition is for* $\mathbb{R}^2$. *The generalization to other dimensions should be obvious.*

Geometrically, one may visualize **v** + **w** by placing the initial point of **w** at the endpoint of **v**. **v** + **w** goes from the initial point of **v** to the endpoint of **w**.

Commutativity

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The Zero Vector

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Every vector $\mathbf{v}$ has an additive inverse, denoted by $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

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Geometrically, if $k > 0$, $k\mathbf{v}$ is a vector in the same direction as $\mathbf{v}$ with a magnitude k times as great.

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Geometrically, if $k > 0$, $k\mathbf{v}$ is a vector in the same direction as $\mathbf{v}$ with a magnitude k times as great.

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Scalar Multiplication

Real numbers are referred to as *scalars*. Multiplication of a vector by a scalar is referred to as scalar multiplication.

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It is easy to see $0\mathbf{v} = \mathbf{0}$ and $1\mathbf{v} = \mathbf{v}$.

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We haven't defined *scalar division*; what we mean is $\dfrac{1}{|\mathbf{v}|} \cdot \mathbf{v}$.

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Law of Cosines

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Using the Law of Cosines and remembering $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$, we have $(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} - 2|\mathbf{v}||\mathbf{w}| \cos\theta$.

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$\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}$.

Similarly,
$\cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}$.
$\cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$.

Projections

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Geometrically, this is the vector along $\mathbf{w}$ whose length is equal to the length of the scalar projection.

Cross Product

The cross product $\mathbf{v} \times \mathbf{w}$ is a vector of length $|\mathbf{v}||\mathbf{w}| \sin\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, such that $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v} \times \mathbf{w}$ form a right-hand triple.

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If the usual rules of algebra, such as the associative and distributive laws, hold for the cross product, we could calculate the cross product of any two vectors by writing them in terms of the standard basis vectors.

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This is a complicated definition. Fortunately, there's a convenient mnemonic device involving symbolic determinants that may be used to calculate cross products.

A matrix is a rectangular array of elements. A square matrix has the same number of rows as columns.

Determinants

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We may think of this as adding the products of elements in each diagonal going down as we go from left to right and subtracting the products of elements in each diagonal going down as we go from right to left.

Cross Product as a Symbolic Determinant

Symbolically, $< a, b, c > \times < d, e, f > = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix}$.

Properties of the Cross Product

One can see $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ by calculating the dot products $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$ and $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w})$.

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For example, if $\mathbf{v} =< x_1, y_1, z_1 >$ and $\mathbf{w} =< x_2, y_2, z_2 >$, then $\mathbf{v} \times \mathbf{w} =< y_1 z_2 - y_2 z_1, x_2 z_1 - x_1 z_2, x_1 y_2 - x_2 y_1 >$, so $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = x_1(y_1 z_2 - y_2 z_1) + y_1(x_2 z_1 - x_1 z_2) + z_1(x_1 y_2 - x_2 y_1) = x_1 y_1 z_2 - x_1 y_2 z_1 + x_2 y_1 z_1 - x_1 y_1 z_2 + x_1 y_2 z_1 - x_2 y_1 z_1 = 0$.

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For example, if $\mathbf{v} =< x_1, y_1, z_1 >$ and $\mathbf{w} =< x_2, y_2, z_2 >$, then $\mathbf{v} \times \mathbf{w} =< y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1 >$, so $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = x_1(y_1z_2 - y_2z_1) + y_1(x_2z_1 - x_1z_2) + z_1(x_1y_2 - x_2y_1) = x_1y_1z_2 - x_1y_2z_1 + x_2y_1z_1 - x_1y_1z_2 + x_1y_2z_1 - x_2y_1z_1 = 0$.

A similar calculation works for $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w})$.

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Noticing the products of squares suggests looking at the product of $|\mathbf{v}|^2 |\mathbf{w}|^2$.

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It follows that $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}| \sin\theta$.

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