Mathematics 211 SOLUTIONS
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Due Wednesday, April 26 This problem set is worth 50 points.

1. Find the general solution of $\dfrac{d^2x}{dt^2} - 16x = 0$.

   Solution: The auxiliary equation is $m^2 - 16 = 0$, which may be factored $(m+4)(m-4) = 0$. It has solutions $m = -4$, $m = 4$, so the differential equation has solutions e^{-4t} and e^{4t} and the general solution is $x = ae^{-4t} + be^{4t}$.

2. Find the general solution of $\dfrac{d^2x}{dt^2} + 16x = 0$.

   Solution: The auxiliary equation is $m^2 + 16 = 0$, which has complex solutions $m = \pm 4i$, so the differential equation has solutions $\cos(4t)$ and $\sin(4t)$ and the general solution is $x = a\cos(4t) + b\sin(4t)$.

3. Find the general solution of $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 4x = 6e^t$.

   Solution: The auxiliary equation is $m^2 - 4m + 4 = 0$, which may be factored to $(m-2)^2 = 0$, so it has a double solution $m = 2$ and the associated homogeneous differential equation has solutions e^{2t} and te^{2t}.

   We can use Judicious Guessing to find a particular solution to the differential equation, guessing $x = ae^t$. Then $\dfrac{dx}{dt} = \dfrac{d^2x}{dt^2} = ae^t$, and plugging into the differential equation we get $ae^t - 4ae^t + 4ae^t = 6e^t$, so $ae^t = 6e^t$ and $a = 6$. So we get $x = 6e^t$ as a particular solution and $x = ae^{2t} + bte^{2t} + 6e^t$ as the general solution.

4. Find the general solution of $\dfrac{d^3x}{dt^3} - 6\dfrac{d^2x}{dt^2} + 11\dfrac{dx}{dt} - 6x = 0$.

   Solution: The auxiliary equation is $m^3 - 6m^2 + 11m - 6 = 0$. Looking at the divisor of 6 for a solution, we get $m = 1$ is a solution, so we factor $m^3 - 6m^2 + 11m - 6 = (m-1)(m^2-5m+6)$. We can finish the factoring by trial and error, getting $m^2-5m+6 = (m-2)(m-3)$, so we can write the auxiliary equation in the form $(m-1)(m-2)(m-3) = 0$.

   Thus the differential equation has solutions e^t, e^{2t} and e^{3t} and general solution $x = ae^t + be^{2t} + ce^{3t}$.

5. A spring is such that a 4 pound weight stretches the spring 0.4 feet. The 4 pound weight is attached to the spring and the weight is started from the equilibrium position with an initial upward velocity of 2 feet per second.

   (a) Set up a differential equation to model this.

   (b) Solve the differential equation.

   (c) Describe the motion of the weight.

   Solution: The force of the spring is given by Hooke's Law, $F = kx$, where F is the force, x is the amount the spring is stretched past the equilibrium position, and k is a constant. We know $x = 0.4$ when $F = 4$, so $4 = k(0.4)$ and thus $k = 10$.

   The appropriate unit of mass is the slug. If we let m be the mass, in slugs, F the weight and g the acceleration due to gravity, $F = mg$, so $m = \dfrac{F}{g}$. Since we have a 4 pound weight and $g \approx 32$, $m \approx \dfrac{4}{32} = \dfrac{1}{8}$.

   Since there is no retarding force, our model becomes
   $\dfrac{1}{8}\dfrac{d^2x}{dt^2} + 10x = 0$, with initial conditions $x(0) = 0$ and $x'(0) = -2$.
   The spring starts at the equilibrium position, so $x(0) = 0$, but it has an initial upward speed of 2 so $\dfrac{dx}{dt} = -2$.

   The auxiliary equation is $\dfrac{1}{8}m^2 + 10 = 0$, which may be solved as follows: $m^2 + 80 = 0$, $m = \pm\sqrt{80}i = \pm 4i\sqrt{5}$.

   The differential equation thus has solutions $\cos(4\sqrt{5}t)$ and $\sin(4\sqrt{5}t)$ and general solution $x = a\cos(4\sqrt{5}t) + b\sin(4\sqrt{5}t)$.

   Since $x(0) = 0$, we get $a = 0$, so $x = b\sin(4\sqrt{5}t)$.

   Since $x' = 4\sqrt{5}b\cos(4\sqrt{5}t)$ and $x'(0) = -2$, we get $4\sqrt{5}b = -2$, $b = -\dfrac{1}{2\sqrt{5}}$, so the solution to the differential equation with initial conditions is $x = -\dfrac{1}{2\sqrt{5}}\sin(4\sqrt{5}t)$.

   The weight starts at the equilibrium position. x goes from 0 to $-\dfrac{1}{2\sqrt{5}}$ to 0 to $\dfrac{1}{2\sqrt{5}}$ back to 0 and then repeats, so the weight moves up $\dfrac{1}{2\sqrt{5}}$ feet above the equilibrium position, then swings back down past the equilibrium position until it's $\dfrac{1}{2\sqrt{5}}$ feet below the equilibrium position, then goes back up to the equilibrium position and repeats the cycle.

   It goes through a full cycle as $4\sqrt{5}t$ increases by 2π, in other words, every $\dfrac{\pi}{2\sqrt{5}} \approx$ 0.702481473 seconds.

6. Repeat the previous question with the added condition that the motion takes place in a medium which furnishes a retarding force of a magnitude numerically equal to the speed of the weight (in feet per second).

   Solution: The differential equation becomes $\frac{1}{8}\frac{d^2x}{dt^2}+\frac{dx}{dt}+10x=0$ with the same initial conditions.

   The auxiliary equation is then $\frac{1}{8}m^2+m+10=0$, which may be solved using the Quadratic Formula. It's easier if we multiply both sides of the equation by 8 before applying the Quadratic Formula:

   $m^2+8m+80=0$, so $m=\frac{-8\pm\sqrt{8^2-4\cdot 1\cdot 80}}{2\cdot 1}=\frac{-8\pm\sqrt{8^2-4\cdot 8\cdot 2\cdot 5}}{2}$
   $=\frac{-8\pm\sqrt{8^2-8^2\cdot 5}}{2}=\frac{-8\pm\sqrt{8^2(1-5)}}{2}=\frac{-8\pm 8\sqrt{-4}}{2}=(-4\pm 8i)$

   We thus get solutions $e^{-4t}\cos(8t)$, $e^{-4t}\sin(8t)$ and the general solution $x=ae^{-4t}\cos(8t)+be^{-4t}\sin(8t)$.

   Since $x(0)=0$, we get $a=0$, so $x=be^{-4t}\sin(8t)$.

   Differentiating, $x'=b(8e^{-4t}\cos(8t)-4e^{-4t}\sin(8t))$. Since $x'(0)=-2$, we get $8b=-2$, so $b=-\frac{1}{4}$ and the solution to the differential equation with the initial conditions is $x=-\frac{1}{4}e^{-4t}\sin(8t)$.

   The motion is similar, but with each cycle the weight swings closer and closer to the equilibrium point.

7. Find the general solution of the system:

$$\frac{dx}{dt} = 4x - y$$
$$\frac{dy}{dt} = 2x + y$$

Solution: We can write the equation in the form $\frac{d\mathbf{X}}{dt} = \mathbf{AX}$, where $A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$.

We get eigenvalues by solving $A - \lambda I = 0$:

$\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0.$

$(4-\lambda)(1-\lambda) - (-1)(2) = 0$

$\lambda^2 - 5\lambda + 4 + 2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 3)(\lambda - 2) = 0$

So the eigenvalues are $\lambda = 3$ and $\lambda = 2$.

We now solve $(A - \lambda I)U = 0$:

For $\lambda = 3$, we get:

$u - v = 0$, $2u - 2v = 0$, so $v = u$. We mav take $u = v = 1$

So we get a solution $X = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = 2$, we get:

$2u - v = 0$, $2u - v = 0$, so $v = 2u$ and we may take $u = 1$, $v = 2$.

So we get a solution $X = e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

The general solution is $X = ae^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + be^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

In scalar form, we have $x = ae^{3t} + be^{2t}$, $y = ae^{3t} + 2be^{2t}$.

8. Use the definition of a Laplace Transform to derive the formula $\mathcal{L}[\sin \omega t] = \dfrac{\omega}{s^2+\omega^2}$.

   Solution: $\mathcal{L}[\sin \omega t] = \int_0^\infty e^{-st} \sin wt\, dt$.

   We could use Integration By Parts to find $I = \int e^{-st} \sin \omega t\, dt$, but we'll use Judicious Guessing, guessing $I = ae^{-st}\sin \omega t + be^{-st}\cos \omega t$.

   Since $I' = (-as - b\omega)e^{-st}\sin \omega t + (a\omega - bs)e^{-st}\cos \omega t$, we get $-as - b\omega = 1$, $a\omega - bs = 0$.

   From the second equation, $b = \dfrac{a\omega}{s}$, so $-as - \dfrac{a\omega}{s}\omega = 1$, $-as^2 - a\omega^2 = s$, $a = -\dfrac{s}{s^2+\omega^2}$.

   Since $b = \dfrac{a\omega}{s}$, it follows that $b = -\dfrac{\omega}{s^2+\omega^2}$.

   We thus have $I = -\dfrac{s}{s^2+\omega^2}e^{-st}\sin \omega t - \dfrac{\omega}{s^2+\omega^2}e^{-st}\cos \omega t$.

   Thus $\mathcal{L}[\sin \omega t] = \int_0^\infty e^{-st}\sin wt\, dt = \lim_{u\to\infty} \int_0^u e^{-st}\sin wt\, dt$

   $= \lim_{u\to\infty} \left[-\dfrac{s}{s^2+\omega^2}e^{-st}\sin \omega t - \dfrac{\omega}{s^2+\omega^2}e^{-st}\cos \omega t \right]_0^t$

   $= \lim_{u\to\infty} -\dfrac{s}{s^2+\omega^2}e^{-su}\sin \omega u - \dfrac{\omega}{s^2+\omega^2}e^{-su}\cos \omega u - (0 - \dfrac{\omega}{s^2+\omega^2}) = -(-\dfrac{\omega}{s^2+\omega^2})$

   $= \dfrac{\omega}{s^2+\omega^2}$

9. Suppose $a, b > 0$. Derive the formula $a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\delta)$, where $\delta = \arccos\left(\dfrac{a}{\sqrt{a^2+b^2}}\right)$.

   Solution: $a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\left(\dfrac{a}{\sqrt{a^2+b^2}}\sin\theta + \dfrac{b}{\sqrt{a^2+b^2}}\cos\theta\right)$

   $= \sqrt{a^2+b^2}(\cos\delta\sin\theta + \sin\delta\cos\theta) = \sqrt{a^2+b^2}\sin(\theta+\delta)$.