1. Find the general solution of $\frac{d^2x}{dt^2} - 16x = 0$.

**Solution:** The auxiliary equation is $m^2 - 16 = 0$, which may be factored $(m+4)(m-4) = 0$. It has solutions $m = -4, m = 4$, so the differential equation has solutions $e^{-4t}$ and $e^{4t}$ and the general solution is $x = ae^{-4t} + be^{4t}$.

2. Find the general solution of $\frac{d^2x}{dt^2} + 16x = 0$.

**Solution:** The auxiliary equation is $m^2 + 16 = 0$, which has complex solutions $m = \pm 4i$, so the differential equation has solutions $\cos(4t)$ and $\sin(4t)$ and the general solution is $x = a\cos(4t) + b\sin(4t)$.

3. Find the general solution of $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 6e^t$.

**Solution:** The auxiliary equation is $m^2 - 4m + 4 = 0$, which may be factored to $(m - 2)^2 = 0$, so it has a double solution $m = 2$ and the associated homogeneous differential equation has solutions $e^{2t}$ and $te^{2t}$.

We can use Judicious Guessing to find a particular solution to the differential equation, guessing $x = ae^t$. Then $\frac{dx}{dt} = \frac{d^2x}{dt^2} = ae^t$, and plugging into the differential equation we get $ae^t - 4ae^t + 4ae^t = 6e^t$, so $ae^t = 6e^t$ and $a = 6$. So we get $x = 6e^t$ as a particular solution and $x = ae^{2t} + bte^{2t} + 6e^t$ as the general solution.

4. Find the general solution of $\frac{d^3x}{dt^3} - 6\frac{d^2x}{dt^2} + 11\frac{dx}{dt} - 6x = 0$.

**Solution:** The auxiliary equation is $m^3 - 6m^2 + 11m - 6 = 0$. Looking at the divisor of 6 for a solution, we get $m = 1$ is a solution, so we factor $m^3 - 6m^2 + 11m - 6 = (m-1)(m^2 - 5m + 6)$. We can finish the factoring by trial and error, getting $m^2 - 5m + 6 = (m-2)(m-3)$, so we can write the auxiliary equation in the form $(m-1)(m-2)(m-3) = 0$.

Thus the differential equation has solutions $e^t$, $e^{2t}$ and $e^{3t}$ and general solution $x = ae^t + be^{2t} + ce^{3t}$. 

This problem set is worth 50 points.
5. A spring is such that a 4 pound weight stretches the spring 0.4 feet. The 4 pound weight is attached to the spring and the weight is started from the equilibrium position with an initial upward velocity of 2 feet per second.

(a) Set up a differential equation to model this.

(b) Solve the differential equation.

(c) Describe the motion of the weight.

Solution: The force of the spring is given by Hooke’s Law, \( F = kx \), where \( F \) is the force, \( x \) is the amount the spring is stretched past the equilibrium position, and \( k \) is a constant. We know \( x = 0.4 \) when \( F = 4 \), so \( 4 = k(0.4) \) and thus \( k = 10 \).

The appropriate unit of mass is the slug. If we let \( m \) be the mass, in slugs, \( F \) the weight and \( g \) the acceleration due to gravity, \( F = mg \), so \( m = \frac{F}{g} \). Since we have a 4 pound weight and \( g \approx 32 \), \( m \approx \frac{4}{32} = \frac{1}{8} \).

Since there is no retarding force, our model becomes

\[ \frac{1}{8} \frac{d^2x}{dt^2} + 10x = 0, \text{ with initial conditions } x(0) = 0 \text{ and } x'(0) = -2. \]

The spring starts at the equilibrium position, so \( x(0) = 0 \), but it has an initial upward speed of 2 so \( \frac{dx}{dt} = -2 \).

The auxiliary equation is \( \frac{1}{8} m^2 + 10 = 0 \), which may be solved as follows: \( m^2 + 80 = 0 \), \( m = \pm \sqrt{80}i = \pm 4i\sqrt{5} \).

The differential equation thus has solutions \( \cos(4\sqrt{5}t) \) and \( \sin(4\sqrt{5}t) \) and general solution \( x = a \cos(4\sqrt{5}t) + b \sin(4\sqrt{5}t) \).

Since \( x(0) = 0 \), we get \( a = 0 \), so \( x = b \sin(4\sqrt{5}t) \).

Since \( x' = 4\sqrt{5}b \cos(4\sqrt{5}t) \) and \( x'(0) = -2 \), we get \( 4\sqrt{5}b = -2 \), \( b = -\frac{1}{2\sqrt{5}} \), so the solution to the differential equation with initial conditions is \( x = -\frac{1}{2\sqrt{5}} \sin(4\sqrt{5}t) \).

The weight starts at the equilibrium position. \( x \) goes from 0 to \( -\frac{1}{2\sqrt{5}} \) to 0 to \( \frac{1}{2\sqrt{5}} \) back to 0 and then repeats, so the weight moves up \( \frac{1}{2\sqrt{5}} \) feet above the equilibrium position, then swings back down past the equilibrium position until it’s \( \frac{1}{2\sqrt{5}} \) feet below the equilibrium position, then goes back up to the equilibrium position and repeats the cycle.

It goes through a full cycle as \( 4\sqrt{5}t \) increases by \( 2\pi \), in other words, every \( \frac{\pi}{2\sqrt{5}} \approx 0.702481473 \) seconds.
6. Repeat the previous question with the added condition that the motion takes place in a medium which furnishes a retarding force of a magnitude numerically equal to the speed of the weight (in feet per second).

**Solution:** The differential equation becomes \( \frac{1}{8} \frac{d^2x}{dt^2} + \frac{dx}{dt} + 10x = 0 \) with the same initial conditions.

The auxiliary equation is then \( \frac{1}{8} m^2 + m + 10 = 0 \), which may be solved using the Quadratic Formula. It’s easier if we multiply both sides of the equation by 8 before applying the Quadratic Formula:

\[
m^2 + 8m + 80 = 0, \quad m = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 1 \cdot 80}}{2} = \frac{-8 \pm \sqrt{64 - 320}}{2} = \frac{-8 \pm \sqrt{-256}}{2} = \frac{-8 \pm 8i}{2} = -4 \pm 4i
\]

We thus get solutions \( e^{-4t} \cos(8t) \), \( e^{-4t} \sin(8t) \) and the general solution \( x = ae^{-4t} \cos(8t) + be^{-4t} \sin(8t) \).

Since \( x(0) = 0 \), we get \( a = 0 \), so \( x = be^{-4t} \sin(8t) \).

Differentiating, \( x' = b(8e^{-4t} \cos(8t) - 4e^{-4t} \sin(8t)) \). Since \( x'(0) = -2 \), we get \( 8b = -2 \), so \( b = -\frac{1}{4} \) and the solution to the differential equation with the initial conditions is \( x = -\frac{1}{4} e^{-4t} \sin(8t) \).

The motion is similar, but with each cycle the weight swings closer and closer to the equilibrium point.
7. Find the general solution of the system:

\[
\begin{align*}
\frac{dx}{dt} &= 4x - y \\
\frac{dy}{dt} &= 2x + y
\end{align*}
\]

**Solution:** We can write the equation in the form \( \frac{dX}{dt} = AX \), where \( A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \).

We get eigenvalues by solving \( A - \lambda I = 0 \):

\[
\begin{vmatrix}
4 - \lambda & -1 \\
2 & 1 - \lambda
\end{vmatrix} = 0.
\]

\((4 - \lambda)(1 - \lambda) - (-1)(2) = 0\)

\(\lambda^2 - 5\lambda + 4 + 2 = 0\)

\(\lambda^2 - 5\lambda + 6 = 0\)

\((\lambda - 3)(\lambda - 2) = 0\)

So the eigenvalues are \( \lambda = 3 \) and \( \lambda = 2 \).

We now solve \( (A - \lambda I)U = 0 \):

For \( \lambda = 3 \), we get:

\(u - v = 0, 2u - 2v = 0\), so \( v = u \). We may take \( u = v = 1 \)

So we get a solution \( X = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

For \( \lambda = 2 \), we get:

\(2u - v = 0, 2u - v = 0\), so \( v = 2u \) and we may take \( u = 1, v = 2 \).

So we get a solution \( X = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

The general solution is \( X = ae^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + be^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

In scalar form, we have \( x = ae^{3t} + be^{2t}, y = ae^{3t} + 2be^{2t} \).
8. Use the definition of a Laplace Transform to derive the formula \( \mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} \).

**Solution:** \( \mathcal{L}[\sin \omega t] = \int_0^\infty e^{-st} \sin \omega t \, dt \).

We could use Integration By Parts to find \( I = \int e^{-st} \sin \omega t \, dt \), but we’ll use Judicious Guessing, guessing \( I = ae^{-st} \sin \omega t + be^{-st} \cos \omega t \).

Since \( I' = (-as - b\omega)e^{-st} \sin \omega t + (a\omega - bs)e^{-st} \cos \omega t \), we get \(-as - b\omega = 1, a\omega - bs = 0\).

From the second equation, \( b = \frac{a\omega}{s} \), so \(-as - \frac{a\omega}{s}\omega = 1, -as^2 - a\omega^2 = s, a = -\frac{s}{s^2 + \omega^2} \).

Since \( b = \frac{a\omega}{s} \), it follows that \( b = -\frac{\omega}{s^2 + \omega^2} \).

Thus \( \mathcal{L}[\sin \omega t] = \int_0^\infty e^{-st} \sin \omega t \, dt = \lim_{u \to \infty} \int_0^u e^{-st} \sin \omega t \, dt 

= \lim_{u \to \infty} \left[ -\frac{s}{s^2 + \omega^2} e^{-st} \sin \omega t - \frac{\omega}{s^2 + \omega^2} e^{-st} \cos \omega t \right]_0^u 

= \lim_{u \to \infty} -\frac{s}{s^2 + \omega^2} e^{-su} \sin \omega u - \frac{\omega}{s^2 + \omega^2} e^{-su} \cos \omega u - (0 - \frac{\omega}{s^2 + \omega^2}) = \left(-\frac{\omega}{s^2 + \omega^2}\right) 

= \frac{\omega}{s^2 + \omega^2} \).

9. Suppose \( a, b > 0 \). Derive the formula \( a \sin \theta + b \cos \theta = \sqrt{a^2 + b^2} \sin(\theta + \delta) \), where \( \delta = \arccos \left( \frac{a}{\sqrt{a^2 + b^2}} \right) \).

**Solution:**

\[
\begin{align*}
 a \sin \theta + b \cos \theta &= \sqrt{a^2 + b^2} \left( \frac{a}{\sqrt{a^2 + b^2}} \sin \theta + \frac{b}{\sqrt{a^2 + b^2}} \cos \theta \right) \\
&= \sqrt{a^2 + b^2} (\cos \delta \sin \theta + \sin \delta \cos \theta) = \sqrt{a^2 + b^2} \sin(\theta + \delta) .
\end{align*}
\]