Mathematics 211 SOLUTIONS
Professor Alan H. Stein
Due Monday, February 27 This problem set is worth 50 points.

1. Set up a differential equation of the form $\frac{dP}{dt} = f(t, P)$, where P represents population and t represents time, assuming a logistic population model, where in the year 2000 the population is $2.5 \cdot 10^8$ (250 million), is growing at an annual rate of $4 \cdot 10^6$ (4 million), and the ecological system can sustain a maximum population of 10^9 (one billion).

   Solution: The differential equation will be of the form $\frac{dP}{dt} = kP(10^9 - P)$, for some $k \in \mathbb{R}$, with initial condition $P = 2.5 \cdot 10^8$ when $t = 2000$.

   We can't get an exact value for k without solving the differential equation, but we can get a good approximation by observing $\frac{d}{dt}(P) \approx 4 \cdot 10^6$ when $P = 2.5 \cdot 10^8$. We can plug those values into the differential equation and then solve for k:

   $4 \cdot 10^6 = k \cdot 2.5 \cdot 10^8(10^9 - 2.5 \cdot 10^8)$, $4 \cdot 10^6 = k \cdot 2.5 \cdot 10^{16}(10 - 2.5)$, $4 \cdot 10^6 = k \cdot 18.75 \cdot 10^{16}$, $4 = k \cdot 1.875 \cdot 10^{11}$, $k = \frac{4}{1.875 \cdot 10^{11}} = \frac{32}{15} \cdot 10^{-11}$.

   So $\frac{dP}{dt} = \frac{32}{15} \cdot 10^{-11} P(10^9 - P)$ is a fairly good model.

(2-10) Consider the differential equation $\frac{dx}{dt} = \frac{x}{25}\left(1 - \frac{x}{10^6}\right)$.

2. Sketch the slope field for the differential equation. *You may submit a graph created by a computer program.*

   Solution:

3. Since this differential equation is autonomous, sketch the phase line.

   Solution:

4. Use the phase line to sketch the solutions of the differential equation. Describe the different classes of solutions in relatively plain language.

   Solution: If x is ever greater than 10^6, then x will decrease and approach 10^6 as $t \to \infty$, while $x \to -\infty$ as t decreases.

   If x is ever between 0 and 10^6, then $x \to 0$ as $t \to \infty$, although $x \to 10^6$ as $t \to -\infty$.

   If x is ever negative, then $x \to \infty$ as t increases and $x \to 0$ as $t \to -\infty$.

5. List the equilibrium points and classify each one as a source, sink or node.

   Solution: The equilibrium points are 0 and 10^6. 0 is a source and 10^6 is a sink.

6. Obtain the general solution for the differential equation.

Solution: The equation is separable and the resulting integral can be obtained using Partial Fractions.

$\frac{dx}{dt} = \frac{x}{25}\left(1 - \frac{x}{10^6}\right)$

$\frac{dx}{dt} = \frac{1}{2.5 \cdot 10^7} \cdot x(10^6 - x)$

$\frac{1}{x(10^6 - x)} \frac{dx}{dt} = \frac{1}{2.5 \cdot 10^7}$

$\int \frac{1}{x(10^6 - x)}\, dx = \int \frac{1}{2.5 \cdot 10^7}\, dt$

Using Partial Fractions, $\frac{1}{x(10^6 - x)} = \frac{a}{x} + \frac{b}{10^6 - x} = \frac{a(10^6 - x) + bx}{10^6 - x}$.

Thus, $ax + b(10^6 - x) = 1$. Letting $x = 0$, we get $10^6 b = 1$, $b = 10^{-6}$.

Letting $x = 10^6$, we also get $10^6 a = 1$, $a = 10^{-6}$.

We thus write the differential equation as

$\int 10^{-6}\left(\frac{1}{x} + \frac{1}{10^6 - x}\right)\, dx = \int \frac{1}{2.5 \cdot 10^7}\, dt$

$\int \frac{1}{x} + \frac{1}{10^6 - x}\, dx = \frac{1}{25} \int 1\, dt$

$\ln|x| - \ln|10^6 - x| = \frac{t}{25} + k$

$\left|\frac{x}{10^6 - x}\right| = \exp(t/25 + k) = \exp(t/25) \cdot e^k$

$\left|\frac{x}{10^6 - x}\right| = c\exp(t/25)$, where $c = e^k > 0$

$x = 10^6 c\exp(t/25) - xc\exp(t/25)$

$x(1 + c\exp(t/25)) = 10^6 c\exp(t/25)$

$x = \frac{10^6 c\exp(t/25)}{1 + c\exp(t/25)}$

$x = \frac{10^6}{1 + \alpha\exp(-t/25)}$, where $\alpha = 1/c > 0$.

(7-10) Consider the same differential equation but with initial condition $x = 8 \cdot 10^6$ when $t = 0$.

7. Find the solution of the differential equation satisfying the given initial conditions.

   Solution:

   Plugging the initial condition into $x = \dfrac{10^6}{1 + \alpha \exp(-t/25)}$, we get $8 \cdot 10^6 = \dfrac{10^6}{1 + \alpha \exp(-4 \cdot 0 \cdot 10^{-14})} = \dfrac{10^6}{1+\alpha}$, so $1 + \alpha = \dfrac{1}{8}$, $\alpha = -\dfrac{7}{8}$, so the solution is $x = \dfrac{10^6}{1 - \dfrac{7}{8}\exp(-t/25)}$.

8. Determine $\lim_{t\to\infty} x$. Explain how this fits in with your answer to question 4.

   Solution: As $t \to \infty$, $\exp(-t/25) \to 0$, so $x \to 10^6$. In the answer to question 4, we inferred $x \to 10^6$ if x was ever bigger than 10^6 and that is consistent with our result here.

9. Use Euler's Method to get a numerical (approximate) solution to the differential equation. Let $\Delta t = 10$ and find $x_0, x_1, x_2, \ldots, x_{10}$.

10. Compare your numerical solution for $t = 100$ to the exact solution. What are the absolute and relative errors?

    Solution: The exact solution gives $x(100) \approx 1,016,287$. The error, both absolute and relative, is far too large for most computers to evaluate.

11. Sketch the phase line for the differential equation $\dfrac{dy}{dt} = y^3 - 16y^2$. Identify each equilibrium point as a skin, source or node.

    Solution: We may factor $\dfrac{dy}{dt} = y^3 - 16y^2 = y^2(y-16) \begin{cases} < 0 & \text{when } y < 0 \\ < 0 & \text{when } 0 < y < 16 \\ > 0 & \text{when } y > 16. \end{cases}$

    The equilibrium points are 0 and 16, with 0 being a node and 16 being a source.

12. Find the general solution for the differential equation $\dfrac{dy}{dx} = \dfrac{2x}{y + x^2 y}$. Find the particular solution satisfying the initial condition $y = 3$ when $x = 2$.

Solution: The solution is mostly a straight calculation.

$\dfrac{dy}{dx} = \dfrac{2x}{y + x^2 y}$

$\dfrac{dy}{dx} = \dfrac{2x}{y(1 + x^2)}$

$y\dfrac{dy}{dx} = \dfrac{2x}{1 + x^2}$

$\int y\, dy = \int \dfrac{2x}{1 + x^2}\, dx$

$\dfrac{y^2}{2} = \ln(1 + x^2) + c$

$y^2 = 2\ln(1 + x^2) + k$

To get the particular solution, plug in $y = 3$, $x = 2$:

$3^2 = 2\ln(1 + 2^2) + k$

$9 = 2\ln 5 + k$

$k = 9 - 2\ln 5$, so $y^2 = 2\ln(1 + x^2) + 9 - 2\ln 5$.

If we want y explicitly, we may take square roots: $y = \pm\sqrt{2\ln(1 + x^2) + 9 - 2\ln 5}$. Since $y = 3 > 0$ when $x = 2$, we take the plus sign and get $y = \sqrt{2\ln(1 + x^2) + 9 - 2\ln 5}$.

(13-16) A room containing 1000 cubic feet of air is originally free of carbon monoxide. The ventilation system starts blowing smoke into the room at a rate of 0.1 cubic feet per minute. The smoke contains a 4 percent concentration of carbon monoxide and is immediately completely mixed with the air in the room, and the well-circulated mixture leaves the room at the same rate the smoke is blown into the room.

13. Set up a model for this situation. Clearly list all relevant variables, along with their meaning. Your model should consist of a differential equation with an initial condition.

    Solution: It's easiest to work with the actual amount of carbon monoxide rather than its concentration. So, let

    $x =$ the amount of carbon monoxide, measured by volume, using cubic feet as the unit,

    $t =$ time, measured in minutes.

    Since the smoke is being blown in at a rate of 0.1 cubic feet per minute and it is 4 percent carbon monoxide, 4% of $0.1 = 0.004$ cubic feet of carbon monoxide is being blown in each minute.

    At any given time, the concentration of carbon monoxide will be $\frac{x}{1000}$, so if 0.1 cubic feet of air is leaving the room per minute, $\frac{x}{1000} \cdot 0.1 = \frac{x}{10000}$ cubic feet of carbon monoxide is leaving per minute.

    The difference between the amount being blown in and the amount leaving gives the rate at which carbon monoxide is changing, so we have $\frac{dx}{dt} = 0.004 - \frac{x}{10000}$.

    Since we start with no carbon monoxide in the room, we get the differential equation

    $$\frac{dx}{dt} = 0.004 - \frac{x}{10000}$$

    with initial condition

    $x = 0$ when $t = 0$.

14. Solve the differential equation you set up.

Solution: We may rewrite the equation as $\frac{dx}{dt} = \frac{40-x}{10^4}$. This is separable, so we may solve as follows:

$\frac{1}{40-x}\frac{dx}{dt} = 10^{-4}$

$\int \frac{1}{40-x}\,dx = 10^{-4}\int dt$

$-\ln|40-x| = 10^{-4}t + k$

Since x starts out being less than 40, it will always be less than 40 and we don't need the absolute value:

$-\ln(40-x) = 10^{-4}t + k$

Plugging in the initial condition:

$-\ln(40-0) = 10^{-4}\cdot 0 + k$

$k = -\ln 40$

We thus have $-\ln(40-x) = 10^{-4}t - \ln 40$

We can solve explicitly for x:

$\ln 40 - \ln(40-x) = 10^{-4}t = t/10000$

$\frac{40}{40-x} = \exp(t/10000)$

$40 - x = 40\exp(-t/10000)$

$x = 40(1-\exp(t/10000)$

15. How long will it take for the concentration of carbon monoxide to reach 1 percent?

Solution: The concentration is 1 percent when $\frac{x}{1000} = 0.01$, so $x = 0.01\cdot 1000 = 10$. So we may plug $x = 10$ into the solution. It's actually easier to plug it into the version $\ln 40 - \ln(40-x) = t/10000$:

$\ln 40 - \ln(40-10) = t/10000$

$\ln 40 - \ln 30 = t/10000$

$t = 10000(\ln 40 - \ln 30) = 10000\ln(4/3) \approx 2876.82072449.$

So it will take approximately 2877 minutes, or about 47 hours, 57 minutes.

16. In the long-term, what level will the concentration of carbon monoxide approach?

Solution:

$\lim_{t\to\infty} x = \lim_{t\to\infty} 40(1-\exp(t/10000) = 40$, so the concentration will approach $\frac{40}{1000} = 0.04 = 4\%$, exactly as might be expected since the smoke being blown in has a concentration of 4%.