Linear Differential Equations With Constant Coefficients

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Homogeneous:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

Non-homogeneous:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

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We'll look at the homogeneous case first and make use of the linear differential operator D.

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The expression
$f(D) = a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \cdots + a_1 D + a_0$ is called a differential operator of order n.

Differential Operators

Given a function y with sufficient derivatives, we define

$$\begin{aligned} f(D)y &= (a_nD^n + a_{n-1}D^{n-1} + a_{n-2}D^{n-2} + \cdots + a_1D + a_0)y \\ &= a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1\frac{dy}{dx} + a_0y \end{aligned}$$

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This gives a convenient way of writing a homogeneous linear differential equation:
$f(D)y = 0$

Properties

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- Commutative Laws
- Associative Laws
- Distributive Law

We can even factor differential operators.

The Auxiliary Equation

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We call $f(m) = 0$ the auxiliary equation.

The Auxiliary Equation: Distinct Roots

If the auxiliary equation $f(m) = 0$ has n distinct roots, $m_1, m_2, m_3, \ldots m_n$, then $e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \ldots, e^{m_n x}$ are distinct solutions of the differential equation $f(D)y = 0$ and the general solution is $c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$.

The Auxiliary Equation: Repeated Roots

Suppose $m = r$ is a repeated root of the auxiliary equation $f(m) = 0$, so that we may factor $f(m) = g(m)(m - r)^k$ for some polynomial $g(m)$ and some integer $k > 1$.

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$$(D-r)^2(xe^{rx}) = (D-r)(D-r)(xe^{rx}) = (D-r)[D(xe^{rx}) - r(xe^{rx})]$$
$$= (D-r)[rxe^{rx} + e^{rx} - rxe^{rx}] = (D-r)e^{rx} = 0$$

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$$(D-r)^3(x^2e^{rx}) = (D-r)^2(D-r)(x^2e^{rx})$$
$$= (D-r)^2[D(x^2e^{rx}) - r(x^2e^{rx})]$$
$$= (D-r)^2[rx^2e^{rx} + 2xe^{rx} - rx^2e^{rx}] = 2(D-r)^2(xe^{rx}) = 0$$

The Auxiliary Equation: Repeated Roots

This type of computation continues through $(D-r)^k(x^{k-1}e^{rx})$, showing $e^{rx}, xe^{rx}, x^2e^{rx}, \ldots x^{k-1}e^{rx}$ are all solutions of the differential equation $f(D)y = 0$.

The Auxiliary Equation: Complex Roots

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$f(m)$ will have factor
$(m - [\alpha + i\beta])(m - [\alpha - i\beta]) = (m - \alpha)^2 + \beta^2$. Thus
$f(D) = g(D)[(D - \alpha)^2 + \beta^2]$ for some operator $g(D)$.

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$(D - \alpha)(e^{\alpha x}\sin(\beta x)) =$
$\beta e^{\alpha x}\cos(\beta x) + \alpha e^{\alpha x}\sin(\beta x) - \alpha e^{\alpha x}\sin(\beta x) = \beta e^{\alpha x}\cos(\beta x)$.

The Auxiliary Equation: Complex Roots

$(D - \alpha)^2(e^{\alpha x} \sin(\beta x))) = (D - \alpha)(D - \alpha)(e^{\alpha x} \sin(\beta x))) =$
$(D - \alpha)[\beta e^{\alpha x} \cos(\beta x)] =$
$\beta[-\beta e^{\alpha x} \sin(\beta x) + \alpha e^{\alpha x} \cos(\beta x) - \alpha e^{\alpha x} \sin(\beta x) = -\beta^2 e^{\alpha x} \sin(\beta x).$

We thus get
$[(D - \alpha)^2 + \beta^2](e^{\alpha x} \sin(\beta x)) = -\beta^2 e^{\alpha x} \sin(\beta x) + \beta^2 e^{\alpha x} \sin(\beta x) = 0.$

Multiple Complex Roots

If $\alpha + i\beta$ occurs as a root with multiplicity $k > 1$, then we get:
$e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x),$
$xe^{\alpha x}\cos(\beta x), xe^{\alpha x}\sin(\beta x),$
$x^2e^{\alpha x}\cos(\beta x), x^2e^{\alpha x}\sin(\beta x),$
$\ldots$
$x^{k-1}e^{\alpha x}\cos(\beta x), x^{k-1}e^{\alpha x}\sin(\beta x)$
are all solutions of the differential equation.

Nonhomogeneous Equations

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So, once we solve the related homogeneous equation, we just have to find one solution of the nonhomogeneous equation.

Solving a Nonhomogeneous Equation

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This is reminiscent of the method of *partial fractions* used to calculate integrals involving rational functions.