Solution to §14.4 #31(b):

Notice that this solution is somewhat slicker than the way we first solved the problem in class. This is typical. You should recognize the value of looking further at a question even after coming up with the answer. Note also that the solutions given often give little insight into how they were originally arrived at.

We set up a coordinate system with the origin at the point along the left bank of the river from where the boat embarks, with the positive x-axis going to the east and the positive y-axis going to the north. We let $\mathbf{r} = \mathbf{r}(t) = (x, y) = (x(t), y(t))$ represent the position of the boat at time t and let $\mathbf{v} = \mathbf{v}(t)$ represent the velocity of the boat.

Suppose the heading of the boat is θ, measured in the standard way. *In other words, θ is the measurement of the angle, measured counterclockwise, that $\mathbf{v}$ would make with the positive x-axis if there was no current.*

Since the boat is traveling at a speed, relative to the water, of 5 meters per second, we would have $\mathbf{v} = (5\cos\theta, 5\sin\theta)$ were it not for the current. Since the current is moving north at a speed of $\frac{3}{400}x(40-x)$, we have $\mathbf{v} = (5\cos\theta, 5\sin\theta + \frac{3}{400}x(40-x))$.

Since the first component of $\mathbf{v}$ is $\dfrac{dx}{dt} = 5\cos\theta$, it follows that $x = 5t\cos\theta + k$ for some constant k. Since $x = 0$ when $t = 0$ (since the boat starts from the origin), it follows that $k = 0$, so $x = 5t\cos\theta$.

We thus get $\mathbf{v} = (5\cos\theta, 5\sin\theta + \frac{3}{400}\cdot 5t\cos\theta(40 - 5t\cos\theta))$.

Simplifying:

$$\begin{aligned}
\mathbf{v} &= 5(\cos\theta, \sin\theta + \frac{3t\cos\theta}{400}(40 - 5t\cos\theta)) \\
&= 5(\cos\theta, \sin\theta + \frac{3t\cos\theta}{400}\cdot 5(8 - t\cos\theta)) \\
&= 5(\cos\theta, \sin\theta + \frac{3t\cos\theta}{80}(8 - t\cos\theta)) \\
\mathbf{v} &= 5(\cos\theta, \sin\theta + \frac{3t\cos\theta}{10} - \frac{3t^2\cos^2\theta}{80})
\end{aligned}$$

We thus get

$$\begin{aligned}
\mathbf{r} &= \int \mathbf{v}(t)\,dt \\
&= 5(t\cos\theta, t\sin\theta + \frac{3t^2\cos\theta}{20} - \frac{t^3\cos^2\theta}{80}) \\
&= 5t(\cos\theta, \sin\theta + \frac{3t\cos\theta}{20} - \frac{t^2\cos^2\theta}{80})
\end{aligned}$$

Since the river is 40 meters wide, $x = 40$ when the boat gets to the other side, so $5t\cos\theta = 40$ and $t\cos\theta = 8$. It follows that at that point in time,

$$\begin{aligned}\mathbf{r} &= 5t(\cos\theta, \sin\theta + \frac{3\cdot 8}{20} - \frac{8^2}{80}) \\ &= 5t(\cos\theta, \sin\theta + \frac{2}{5})\end{aligned}$$

Since we want the boat to wind up directly across the river from its starting point, we need $\sin\theta + \frac{2}{5} = 0$, $\sin\theta = -\frac{2}{5}$, $\theta = \arcsin(-\frac{2}{5}) \approx -0.411516846 \approx -23.5781784782°$.

Since $\cos^2\theta + \sin^2\theta = 1$, $\cos^2\theta + (-\frac{2}{5})^2 = 1$, $\cos^2\theta + \frac{4}{25} = 1$, $\cos^2\theta = \frac{21}{25}$, $\cos\theta = \pm\frac{\sqrt{21}}{5}$. Since $-\pi/2 \leq \theta \leq \pi/2$, $\cos\theta$ must be non-negative, so $\cos\theta = \frac{\sqrt{21}}{5}$.

We thus get

$$\begin{aligned}\mathbf{r} &= 5t(\frac{\sqrt{21}5}{,} - \frac{2}{5} + \frac{3t(\frac{\sqrt{21}}{5})}{20} - \frac{t^2(\frac{\sqrt{21}}{5})^2}{80}) \\ &= 5t(\frac{\sqrt{21}}{5}, -\frac{2}{5} + \frac{3t\sqrt{21}}{100} - \frac{21t^2}{2000})\end{aligned}$$

Note we may find the y component in terms of the x component. If we do that, we find y is a cubic function of x. This is consistent with our intuition regarding the path of the boat.