Mathematics 210 SOLUTIONS
Professor Alan H. Stein
Due Monday, November 26, 2007 This problem set is worth 50 points.

1. Consider $\iiint_{\mathcal{D}} x^2+3xz\,dV$, where $\mathcal{D}$ is the solid bounded by the graphs of $x^2+y^2-z=-1$ and $x^2+y^2+z=37$. Sketch the region $\mathcal{D}$ and write the triple integral as an iterated integral using rectangular coordinates.

   Solution: $x^2+y^2-z=-1$ may be written in the form $z=1+(x^2+y^2)$. It's graph is clearly a paraboloid, facing up, with vertex $(0,0,1)$.

   $x^2+y^2+z=37$ may be written in the form $z=37-(x^2+y^2)$. It's graph is clearly a paroboloid, facing down, with vertex $(0,0,37)$. They would appear to intersect in a circle, which is confirmed if one eliminates z from the equations: $1+(x^2+y^2)=37-(x^2+y^2)$, $2(x^2+y^2)=36$, $x^2+y^2=18$. Thus, the solid lies over the circle in the xy-plane with center at the origin and radius $\sqrt{18}=3\sqrt{2}$.

   We can solve $x^2+y^2=18$ for y, getting $y=\pm\sqrt{18-x^2}$.

   We thus can write $\iiint_{\mathcal{D}} x^2+3xz\,dV=\int_{-3\sqrt{2}}^{3\sqrt{2}}\,dx\int_{-\sqrt{18-x^2}}^{\sqrt{18-x^2}}\,dy\int_{1+(x^2+y^2)}^{37-(x^2+y^2)}\,dz(x^2+3xz)$.

2. Write the triple integral in the previous question as an iterated integral using cylindrical coordinates.

   Solution: Using cylindrical coordinates, $x^2+y^2=r$, so the surfaces become $z=1+r^2$ and $z=37-r^2$. We also have $x=r\cos\theta$ and $y=r\sin\theta$ and we can write
   $\iiint_{\mathcal{D}} x^2+3xz\,dV=\int_0^{2\pi}\,d\theta\int_0^{3\sqrt{2}}\,dr\int_{1+r^2}^{37-r^2}\,dz\,r[(r\cos\theta)^2+3r\cos\theta\cdot z)]$
   $=\int_0^{2\pi}\,d\theta\int_0^{3\sqrt{2}}\,dr\int_{1+r^2}^{37-r^2}\,dz\,r^2\cos\theta(r\cos\theta+3z)$.

3. William Tell's son places a perfectly spherical apple with a radius of 5 centimeters on his head. William Tell shoots an arrow with a radius of a half centimeter directly through the center of the apple. What is the volume of the remaining portion of the apple?

   Solution: Place the apple with center at the origin and the core vertical. In cylindrical coordinates, the surface of the apple has equation $r^2+z^2=25$,
   so $-\sqrt{25-r^2}\le z\le\sqrt{25-r^2}$. In the portion that's left over, $\frac{1}{2}\le r\le 5$.

   The volume is $\iiint_{\text{remaining portion}}\,dV=\int_0^{2\pi}\,d\theta\int_{\frac{1}{2}}^{5}\,dr\int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}}\,dzr$
   $=\int_0^{2\pi}\,d\theta\int_{\frac{1}{2}}^{5}\,drrz\Big|_{z=-\sqrt{25-r^2}}^{z=\sqrt{25-r^2}}=\int_0^{2\pi}\,d\theta\int_{\frac{1}{2}}^{5}\,dr2r\sqrt{25-r^2}$
   $=\int_0^{2\pi}\,d\theta\big[-\frac{2}{3}(25-r^2)^{3/2}\big]_{\frac{1}{2}}^{5}=\int_0^{2\pi}\,d\theta\frac{2}{3}(99/4)^{3/2}=2\pi\cdot\frac{2}{3}(3\sqrt{11}/2)^3=\dfrac{99\pi\sqrt{11}}{2}$.

4. Consider a tetrahedron with vertices $(1,0,0)$, $(0,1,0)$, $(-1,0,0)$, $(0,-1,0)$ and $(0,0,2)$ and density $\delta(x,y,z) = |x| + |y| + |z|$. Sketch the tetrahedron and find its volume, mass, all three first moments and its center of mass. *It should be obvious that two of the moments and two of the coordinates of the center of mass will equal 0, but just saying it's obvious will obviously not suffice.*

   Solution: The region may be describes as $\{(x,y,z) : 0 \le z \le 2 - 2|x| - 2|y|,$ $|x| - 1 \le y \le 1 - |x|,\ -1 \le x \le 1\}$.

   The volume is $V = \iint_{\text{tetrahedron}} dV = \int_{-1}^{1} dx \int_{|x|-1}^{1-|x|} dy \int_{0}^{2-2|x|-2|y|} dz$.

   From the symmetry, it is clear we can take the volume of the portion above the first quadrant and quadruple it, enabling us to eliminate the absolute values and write $V = 4\int_0^1 dx \int_0^{1-x} dy \int_0^{2-2x-2y} dz = 4\int_0^1 dx \int_0^{1-x} dyz\Big|_0^{2-2x-2y}$
   $= 4\int_0^1 dx \int_0^{1-x} dy(2-2x-2y) = 4\int_0^1 dx\big[2y - 2xy - y^2\big]_0^{1-x}$
   $= 4\int_0^1 dx(1-x)^2 = -4(1-x)^3/3\Big|_0^1 = 4/3$. *Note this is exactly* $1/3$ *the volume of the cube whose base is the same as the base of the tetrahedron.*

   We can use symmetry the same way in getting the mass m, obtaining
   $m = 4\int_0^1 dx \int_0^{1-x} dy \int_0^{2-2x-2y} dz(x+y+z) = 4\int_0^1 dx \int_0^{1-x} dy\big[xz + yz + z^2/2\big]_0^{2-2x-2y} =$ $4\int_0^1 dx \int_0^{1-x} dy2(1-x-y) = 4\int_0^1 dx\big[2y - 2xy - y^2\big]_0^{1-x} = 4\int_0^1 dx(1-x)^2 = 4/3$. *The end of the calculation is identical with the end of the calculation for volume!*

   From the symmetry, it's clear the moments about both the xz and yz planes will be 0, so we will just calculate the moment m_{xy} about the xy plane. Once again, we can use symmetry and just deal with the portion above the first quadrant and quadruple it.
   $m_{xy} = 4\int_0^1 dx \int_0^{1-x} dy \int_0^{2-2x-2y} dzz(x+y+z) =$
   $4\int_0^1 dx \int_0^{1-x} dy\big[xz^2/2 + yz^2/2 + z^3/3\big]_0^{2-2x-2y} = \frac{8}{3}\int_0^1 dx \int_0^{1-x} dy(1-x-y)^2(4-x-y)$.
   This may be multiplied out and then integrated term by term, but I'll make use of the following: $(1-x-y)^2(4-x-y) = (1-x-y)^2[3+(1-x-y)] = 3(1-x-y)^2+(1-x-y)^3$, so $m_{xy} = \frac{8}{3}\int_0^1 dx \int_0^{1-x} dy[3(1-x-y)^2 + (1-x-y)^3] =$
   $\frac{8}{3}\int_0^1 dx\big[-(1-x-y)^3 - (1-x-y)^4/4\big]_0^{1-x} = \frac{8}{3}\int_0^1 dx[(1-x)^3 + (1-x)^4/4] =$
   $\frac{8}{3}\big[-(1-x)^4/4 - (1-x)^5/20\big]_0^1 = \frac{8}{3}(1/4 + 1/20) = 4/5$.

   If we let $(\overline{x}, \overline{y}, \overline{z})$ be the center of mass, we have $\overline{z} = \dfrac{m_{xy}}{m} = \dfrac{4/5}{4/3} = \dfrac{3}{5}$, so the center of mass is $(0,0,3/5)$.

5. Calculate $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$ if $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$.

Solution: $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \det\begin{pmatrix} \dfrac{\partial x}{\partial\rho} & \dfrac{\partial x}{\partial\theta} & \dfrac{\partial x}{\partial\phi} \\ \dfrac{\partial y}{\partial\rho} & \dfrac{\partial y}{\partial\theta} & \dfrac{\partial y}{\partial\phi} \\ \dfrac{\partial z}{\partial\rho} & \dfrac{\partial z}{\partial\theta} & \dfrac{\partial z}{\partial\phi} \end{pmatrix}$

$= \det\begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix}$

Factoring out $\rho\sin\phi$ from the second column and ρ from the third column yields:

$\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2\sin\phi\ \det\begin{pmatrix} \sin\phi\cos\theta & -\sin\theta & \cos\phi\cos\theta \\ \sin\phi\sin\theta & \cos\theta & \cos\phi\sin\theta \\ \cos\phi & 0 & -\sin\phi \end{pmatrix} =$

$\rho^2\sin\phi\big(\sin\phi\cos\theta\det\begin{pmatrix} \cos\theta & \cos\phi\sin\theta \\ 0 & -\sin\phi \end{pmatrix} - (-\sin\theta)\det\begin{pmatrix} \sin\phi\sin\theta & \cos\phi\sin\theta \\ \cos\phi & -\sin\phi \end{pmatrix}$

$+ \cos\phi\cos\theta\det\begin{pmatrix} \sin\phi\sin\theta & \cos\theta \\ \cos\phi & 0 \end{pmatrix}\big) =$

$\rho^2\sin\phi[sin\phi\cos\theta(-\cos\theta\sin\phi) + \sin\theta(-\sin^2\phi\sin\theta - \cos^2\phi\sin\theta) + \cos\phi\cos\theta(-\cos\phi\cos\theta)]$

$= -\rho^2\sin\phi[\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi\sin^2\theta + \cos^2\phi\cos^2\theta]$

$= -\rho^2\sin\phi(\sin^2\phi + \cos^2\phi)(\sin^2\theta + \cos^2\theta) = -\rho^2\sin\phi.$

Note, if we had switched θ and ϕ and calculated $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}$ instead, we would have obtained $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin\phi$.

6. Let $\mathcal{D}$ be the region bounded by the x-axis and the lines $x-y=2$, $y=1$ and $x=y$. Sketch $\mathcal{D}$. Calculate $\iint_{\mathcal{D}} x^2+2xy\,dA$ using an iterated integral with x and y as the variables of integration. Recalculate the integral using the change of variables $x=2u+v$, $y=v$. *Obviously, the value of the integral shouldn't change.*

   Solution: The region is a parallelogram. It is easily seen the horizontal lines intersect the line $x=y$ at the points $(0,0)$ and $(1,1)$, while they intersect the line $x-y=2$ at the points $(2,0)$ and $(3,1)$. Those four points are the vertices of the parallelogram. It's easiest to integrate with respect to x first, since then the region does not have to be split up. *If one integrates with respect to y first, one must split the region into three parts: $0\le x\le 1$, $1\le x\le 2$, $2\le x\le 3$.*

   We obtain $\iint_{\mathcal{D}} x^2+2xy\,dA = \int_0^1 dy \int_y^{y+2} dx x^2+2xy = \int_0^1 dy\big[x^3/3+x^2y\big]_y^{y+2}$
   $= \int_0^1 6y^2+8y+\frac{8}{3}\,dy = 2y^3+4y^2+\frac{8}{3}y\Big|_0^1 = 2+4+\frac{8}{3}-0=\frac{26}{3}$.

   With the change of variables, $\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{pmatrix} = \det\begin{pmatrix} 2 & 1 \\ 0 & 1\end{pmatrix} = 2.$

   To find the boundaries of the region, we may take their equations and rewrite them in terms of u and v.

   $x-y=2$ becomes $(2u+v)-v=2$, $2u=2$, $u=1$
   $x=y$ becomes $2u+v=v$, $2u=0$, $u=0$
   $y=1$ becomes $v=1$
   The x-axis $y=0$ becomes $v=0$.

   We thus get a parallelogram with vertices $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$.

   $\iint_{\mathcal{D}} x^2+2xy\,dA = \iint_{\mathcal{D}'}(x^2+2xy)\dfrac{\partial(x,y)}{\partial(u,v)}\,du\,dv = 2\iint_{\mathcal{D}'} 4u^2+8uv+3v^2\,du\,dv$
   $= 2\int_0^1 du\int_0^1 dv(4u^2+8uv+3v^2) = 2\int_0^1 du\big[4u^2v+4uv^2+v^3\big]_0^1 = 2\int_0^1 4u^2+4u+1\,du$
   $= 2\big[\frac{4}{3}u^3+2u^2+u\big]_0^1 = 2(\frac{4}{3}+2+1-0)=\frac{26}{3}$.