Mathematics 210 — SOLUTIONS
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Due Monday, October 29, 2007 — This problem set is worth 50 points.

1. Find an equation tangent to the surface $x^2 + 5xyz + y^2 + z^3 = 80$ at the point $(2, -1, 5)$.

   Solution: $\bigtriangledown(x^2 + 5xyz + y^2 + z^3)\big|_{(2,-1,5)} = (2x + 5yz, 5xz + 2y, 5xy + 3z^2)\big|_{(2,-1,5)} = (-21, 48, 65)$, so an equation of the tangent plane is given by $(-21, 48, 65) \cdot (x, y, z) = (-21, 48, 65) \cdot (2, -1, 5)$ or $-21x + 48y + 65z = 235$.

2. Find all critical points for the function $f(x, y) = (1-x)(1-y)(x+y-1)$ and determine whether they are relative extrema or saddle points.

   Solution: $f_x = (1 - y)[(1 - x) \cdot 1 + (x + y - 1) \cdot (-1)] = (1 - y)(2 - 2x - y)$

   $f_y = (1 - x)(2 - x - 2y)$.

   We thus need to solve:

   $$(1 - y)(2 - 2x - y) = 0$$
   $$(1 - x)(2 - x - 2y) = 0$$

   From the first equation, either $y = 1$ or $y = 2 - 2x$.

   From the second equation, either $x = 1$ or $x = 2 - 2y$.

   One solution is thus $x = 1$, $y = 1$.

   If $y = 1$ and $x = 2 - 2y$, we get the solution $x = 0$, $y = 1$.

   If $x = 1$ and $y = 2 - 2x$, we get the solution $x = 1$, $y = 0$.

   If $y = 2-2x$ and $x = 2-2y$, we have $x = 2-2(2-2x)$, $x = 4x-2$, $3x = 2$, $x = \frac{2}{3}$. We then get $y = 2 - 2 \cdot \frac{2}{3} = \frac{2}{3}$.

   So we have four critical points: $(0, 1)$, $(1, 0)$, $(1, 1)$, $(\frac{2}{3}, \frac{2}{3})$.

   $f_{xx} = (1 - y)(-2) = 2(y - 1)$.

   $f_{xy} = f_{yx} = (1 - y)(-1) + (2 - 2x - y)(-1) = 2x + 2y - 3$.

   $f_{yy} = (1 - x)(-2) = 2(x - 1)$.

   $D = f_{xy}^2 - f_{xx}f_{yy} = (2x + 2y - 3)^2 - 4(x - 1)(y - 1)$.

   At $(0, 1)$, $(1, 0)$ and $(1, 1)$, D is obviously positive, so each is a saddle point.

   At $(\frac{2}{3}, \frac{2}{3})$, $D = \left(-\frac{1}{3}\right)^2 - 4\left(-\frac{1}{3}\right)^2 < 0$, so f has an extremum. Since $f_{xx} = -\frac{2}{3} < 0$, the extremum is a maximum.

3. Find all critical points for the function $f(x,y) = x^4 - 2(x-y)^2 + y^4$ and determine whether they are relative extrema or saddle points.

   Solution: $f_x = 4x^3 - 4(x-y) = 4(x^3 - x + y)$.

   $f_y = 4(x-y) + 4y^3 = 4(x + y^3 - y)$.

   At a critical point, we need $f_x = f_y = 0$, so we obviously need $x^3 - x + y = 0$ and $x + y^3 - y = 0$. It follows that $(x^3 - x + y) + (x + y^3 - y) = 0$, so $x^3 + y^3 = 0$. It follows that $y = -x$.

   We thus have $x^3 - x - x = 0$, $x^3 - 2x = 0$, $x(x^2 - 2) = 0$, so either $x = 0$ or $x = \pm\sqrt{2}$.

   We thus have critical points $(0,0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$.

   $f_{xx} = 4(3x^2 - 1)$, $f_{xy} = 4$, $f_{yy} = 4(3y^2 - 1)$.

   $D = f_{xy}^2 - f_{xx}f_{yy} = 4^2 - 4(3x^2 - 1) \cdot 4(3y^2 - 1) = 16[1 - (3x^2 - 1)(3y^2 - 1)]$.

   At $(0,0)$, $D = 0$ so this test in inconclusive.

   At each of the other points, $D = 16[1 - 5^2] < 0$, so we have an extremum. At each of those points, $f_{xx} = 4(3 \cdot 2 - 1) = 20 > 0$, so each in a minimum.

   Note: we may examine the values of f near the origin more closely. Note $f(x,x) = 2x^4 > 0 = f(0,0)$ for x near 0, while $f(x,-x) = 2x^4 - 2(2x)^2 = 2x^4 - 8x^2 = 2x^2(x^2 - 4) < 0 = f(0,0)$ for $|x| < 2$, so the origin is actually a saddle point.

4. Find the points on the ellipsoid $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{25} = 1$ closest to and farthest from the origin.

   Solution: If we let $f(x,y,z) = x^2 + y^2 + z^2$ and $g(x,y,z) = \dfrac{x^2}{4} + y^2 + \dfrac{z^2}{25}$, we need to find the extreme values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = 1$.

   Since $\bigtriangledown f = (2x, 2y, 2z) = 2(x,y,z)$ and $\bigtriangledown g = \left(\frac{2x}{4}, 2y, \frac{2z}{25}\right) = \frac{1}{50}(25x, 100y, 4z)$, any extremum must occur at a point where $(x,y,z) = \lambda(25x, 100y, 4z)$ and $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{25} = 1$.

   We must therefore solve: $x = 25\lambda x$, $y = 100\lambda y$, $z = 4\lambda z$, $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{25} = 1$.

   If $x \neq 0$, then $\lambda = \frac{1}{25}$, while if $y \neq 0$ then $\lambda = \frac{1}{100}$ and if $z \neq 0$, then $\lambda = \frac{1}{4}$. Thus, at least two of x, y, z must be 0, while all three cannot be 0 since the origin is not on the ellipsoid.

   The extreme values must therefore occur at the points where the ellipsoid intersects the coordinate axes. Those points are obviously:

   $(\pm 2, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 5)$.

   Obviously, then, $(0,1,0)$ and $(0,-1,0)$ are the closest points and $(0,0,5)$ and $(0,0,-5)$ are the farthest points.

5. Find the shortest distance between the circle $x^2+y^2 = 1$ and the curve $x^2y = 16$. *Getting to the point where you need to solve a system of equations in order to find possible points where the curves are closest will be enough to earn full credit; actually finding the shortest distance will earn extra credit.*

Solution: Consider a point (x, y) on the circle and a point (u, v) on the other curve. The closest distance between the circle and the curve will be the square root of the minimum value of $f(x, y, u, v) = (x - u)^2 + (y - v)^2$, subject to the constraints $x^2 + y^2 = 1$ and $u^2v = 16$.

If we let $g(x, y, u, v) = x^2 + y^2$ and $h(x, y, u, v) = u^2v$, we can use Lagrange Multipliers and solve:

$\bigtriangledown f = \lambda \bigtriangledown g + \mu \bigtriangledown h$
$g(x, y, u, v) = 1$
$h(x, y, u, v) = 16.$

Since
$\bigtriangledown f =< 2(x - u), 2(y - v), -2(x - u), -2(y - v) >,$
$\bigtriangledown g =< 2x, 2y, 0, 0 >$ and
$\bigtriangledown h =< 0, 0, 2uv, u^2 >,$

we need to solve:
$2(x - u) = \lambda \cdot 2x$
$2(y - v) = \lambda \cdot 2y$
$-2(x - u) = \mu \cdot 2uv$
$-2(y - v) = \mu \cdot u^2$
$x^2 + y^2 = 1$
$u^2v = 16.$

These easily simplify slightly to:

$x - u = \lambda x$
$y - v = \lambda y$
$x - u = -\mu uv$
$y - v = -\frac{1}{2}\mu u^2$
$x^2 + y^2 = 1$
$u^2v = 16$

From the first and third equations we get $\lambda x = -\mu uv$.

From the second and fourth equations we get $\lambda y = -\frac{1}{2}\mu u^2$.

Hence, $\dfrac{\lambda y}{\lambda x} = \dfrac{-\frac{1}{2}\mu u^2}{-\mu uv}$, so
$\dfrac{y}{x} = \dfrac{1}{2}\dfrac{u}{v}$.
From the first two equations, we can get: $(x-u)\lambda y = (y-v)\lambda x$, so $(x-u)y = (y-v)x$, $xy - uy = yx - vx$, $-uy = -vx$, $\dfrac{y}{x} = \dfrac{v}{u}$.

We thus have $\frac{1}{2}\frac{u}{v} = \frac{v}{u}$, $\left(\frac{u}{v}\right)^2 = 2$, $\frac{u}{v} = \pm\sqrt{2}$, $u = v\sqrt{2}$.

Since $u^2v = 16$, we have $(v\sqrt{2})^2 v = 16$, $2v^3 = 16$, $v^3 = 8$, $v = 2$ and thus $u = \pm 2\sqrt{2}$.

Since $\frac{y}{x} = \frac{v}{u} = \frac{1}{u/v}$, we also have $\frac{y}{x} = \pm\frac{1}{\sqrt{2}}$ and $x = \pm y\sqrt{2}$. Since $x^2 + y^2 = 1$, we have $(\pm y\sqrt{2})^2 + y^2 = 1$, $2y^2 + y^2 = 1$, $3y^2 = 1$, $y^2 = \frac{1}{3}$, $y = \pm\frac{1}{\sqrt{3}}$ and $x = \pm y\sqrt{2} = \pm\sqrt{\frac{2}{3}}$.

Since $x^2y = 16$ is symmetric with respect to the y-axis, above the circle and the graph falls totally in the upper half plane, there clearly will be two places where the curves are closest, once in the first quadrant and once in the second.

In the first quadrant, we have $(x, y) = (\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}})$ and $(u, v) = (2\sqrt{2}, 2)$, so $f(x, y, u, v) = (\sqrt{\frac{2}{3}} - 2\sqrt{2})^2 + (\frac{1}{\sqrt{3}} - 2)^2 = 2(\frac{1}{\sqrt{3}} - 2)^2 + (\frac{1}{\sqrt{3}} - 2)^2 = 3(\frac{1}{\sqrt{3}} - 2)^2$.

The closest distance is thus $\sqrt{3}(2 - \frac{1}{\sqrt{3}}) = 2\sqrt{3} - 1$.

6. Use a double integral to find the area of the region in the first quadrant between $x = 1$ and $x = 2$ and beneath the curve $y = \frac{1}{x}$.

   Solution: Call the region $\mathcal{D}$. The area $= \iint_{\mathcal{D}} dA = \int_1^2 dx \int_0^{1/x} dy = \int_1^2 dx y\Big|_0^{1/x} = \int_1^2 dx(1/x) = \int_1^2 \frac{1}{x}\, dx = \ln 2$ (by the definition of the natural logarithm function).

(7-11): Rewrite each double integral as an iterated integral.

7. $\iint_{\mathcal{D}} x^2 + 3xy\, dA$, where $\mathcal{D} = \{(x, y) | 2x \le y \le 5x + 3,\ 0 \le x \le 10\}$.

   Solution: $\iint_{\mathcal{D}} x^2 + 3xy\, dA = \int_0^{10} dx \int_{2x}^{5x+3} dy(x^2 + 3xy)$.

8. $\iint_{\mathcal{D}} x^2 + 3xy\, dA$, where $\mathcal{D}$ is the circle with center at $(-3, 5)$ and radius 2. *Set up the iterated integral so that the integration is done first with respect to y.*

   Solution: The equation of the circle may be written in the form $(x+3)^2 + (y-5)^2 = 4$. Solving for y, we get $(y - 5)^2 = 4 - (x + 3)^2$, $y - 5 = \pm\sqrt{4 - (x + 3)^2}$,
   $y = 5 \pm \sqrt{4 - (x + 3)^2}$.

   We thus have $\iint_{\mathcal{D}} x^2 + 3xy\, dA = \int_{-5}^{-1} dx \int_{5-\sqrt{4-(x+3)^2}}^{5+\sqrt{4-(x+3)^2}} dy(x^2 + 3xy)$.

9. $\iint_{\mathcal{D}} x^2 + 3xy\, dA$, where $\mathcal{D}$ is the circle with center at $(-3, 5)$ and radius 2. *Set up the iterated integral so that the integration is done first with respect to x.*

   Solution: The equation of the circle may be written in the form $(x+3)^2 + (y-5)^2 = 4$. Solving for x, we get $(x + 3)^2 = 4 - (y - 5)^2$, $x + 3 = \pm\sqrt{4 - (y - 5)^2}$,
   $x = -3 \pm \sqrt{4 - (y - 5)^2}$.

   We thus have $\iint_{\mathcal{D}} x^2 + 3xy\, dA = \int_3^7 dy \int_{-3-\sqrt{4-(y-5)^2}}^{-3+\sqrt{4-(y-5)^2}} dx(x^2 + 3xy)$.

10. $\iint_{\mathcal{D}} x^2 + 3xy\, dA$, where $\mathcal{D}$ is the region bounded by the lines $x+y=7$, $5x+2y=29$ and $y=2x+1$. *Set up the iterated integral so that the integration is done first with respect to y.*

    Solution: We need to find where each pair of lines intersects. The region bounded by th lines will be the triangle with those three points as its vertices. To find the points of intersection, we solve each pair of equations simultaneously.

    To solve $x+y=7$, $5x+2y=29$, we have $y=7-x$, $5x+2(7-x)=29$, $5x+14-2x=29$, $3x=15$, $x=5$, $y=2$. So the point of intersection is $(5,2)$.

    To solve $x+y=7$, $y=2x+1$, we have $x+(2x+1)=7$, $3x+1=7$, $3x=6$, $x=2$, $y=5$. So their point of intersection is $(2,5)$.

    To solve $5x+2y=29$, $y=2x+1$, we have $5x+2(2x+1)=29$, $5x+4x+2=29$, $9x=27$, $x=3$, $y=7$. So their point of intersection is $(3,7)$.

    To integrate with respect to y first, we need to divide the region at $x=3$.
    For $2 \le x \le 3$, the bottom of the region is the line $x+y=7$, which may be written as $y=7-x$, and the top is the line $y=2x+1$.
    For $3 \le x \le 5$, the bottom is the same as before but the top is the line $5x+2y=29$, which may be written as $y=\frac{29-5x}{2}$.

    We thus get $\iint_{\mathcal{D}} x^2+3xy\, dA = \int_2^3 dx \int_{7-x}^{2x+1} dy(x^2+3xy) + \int_3^5 dx \int_{7-x}^{\frac{29-5x}{2}} (x^2+3xy)$

11. $\iint_{\mathcal{D}} x^2 + 3xy\, dA$, where $\mathcal{D}$ is the region bounded by the lines $x+y=7$, $5x+2y=29$ and $y=2x+1$. *Set up the iterated integral so that the integration is done first with respect to x.*

    Solution: To integrate with respect to x first, we must divide the region at $y=5$.
    For $2 \le y \le 5$, the region is bounded on the left by the line $x+y=7$, which may be written as $x=7-y$, and on the right by $5x+2y=29$, which may be written as $x=\frac{29-2y}{5}$.
    For $5 \le y \le 7$, the region is bounded on the left by the line $y=2x+1$, which may be written as $x=\frac{y-1}{2}$, and on the right as before, by the line $x=\frac{29-2y}{5}$.

    We thus get $\iint_{\mathcal{D}} x^2+3xy\, dA = \int_2^5 dy \int_{7-y}^{\frac{29-2y}{5}} (x^2+3xy) + \int_5^7 dy \int_{\frac{y-1}{2}}^{\frac{29-2y}{5}} (x^2+3xy)$.