Mathematics 210 SOLUTIONS
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Monday, October 1, 2007 This problem set is worth 50 points.

1. Find the angle between the vectors $(2, 7, 3)$ and $(5, -2, 4)$.

   Solution: Let θ be the angle. $(2, 7, 3) \cdot (5, -2, 4) = |(2, 7, 3)| \cdot |(5, -2, 4)| \cos\theta$, so $8 = \sqrt{62}\sqrt{45}\cos\theta$, $\cos\theta = \dfrac{8}{3\sqrt{310}}$, so $\theta = \arccos\left(\dfrac{8}{3\sqrt{310}}\right)$.

2. Find vector parametric, scalar parametric and scalar symmetric equations for the line through the point $(5, 3, 1)$ and orthogonal to the plane $3x - 7y + 2z = 8$.

   Solution: The normal to the plane is in the direction of the line, so the vector $(3, -7, 2)$ is in the direction of the line.

   Vector Equation: $\mathbf{x} = (5, 3, 1) + (3, -7, 2)t$.

   Scalar Parametric Equations: $x = 5 + 3t$, $y = 3 - 7t$, $z = 1 + 2t$.

   Scalar Symmetric Equations: $\dfrac{x-5}{3} = \dfrac{y-3}{-7} = \dfrac{z-1}{2}$.

3. Find an equation of the plane through the points $(1, 5, 4)$, $(4, 9, 1)$ and $(5, 17, 5)$. *Extra Credit: Obtain the same equation using an entirely di erent method.*

   Solution: Let $\mathbf{v} = (4, 9, 1) - (1, 5, 4) = (3, 4, -3)$, $\mathbf{w} = (5, 17, 5) - (1, 5, 4) = (4, 12, 1)$, $\mathbf{n} = \mathbf{v} \times \mathbf{w} = (3, 4, -3) \times (4, 12, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -3 \\ 4 & 12 & 1 \end{vmatrix} = (40, -15, 20) = 5(8, -3, 4)$. Since we can use any vector in the same direction as a normal to the plane, we redefine $\mathbf{n} = (8, -3, 4)$. We get an equation for the plane $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot (1, 5, 4)$. Writing $\mathbf{x} = (x, y, z)$, we get $(8, -3, 4) \cdot (x, y, z) = (8, -3, 4) \cdot (1, 5, 4)$ or $8x - 3y + 4z = 9$

   Alternatively, we know the three points satisfy an equation of the form $ax + by + cz = d$. We may thus try to solve the following system for a, b, c, d.

$$\begin{aligned} a(1) + b(5) + c(4) &= d \\ a(4) + b(9) + c(1) &= d \\ a(5) + b(17) + c(5) &= d \end{aligned}$$

   Since we have an extra variable, if we so desire we can arbitarily choose a value for one of them. If we let $d = 9$, we will obtain $a = 8$, $b = -3$, $c = 4$. If we chose any other value for d, we would have obtained an equivalent equation.

4. Find the distance between the point $(1, 2, 3)$ and the plane $4x + 5y + 6z = 7$. *Extra Credit: Find the distance using an entirely di erent method.*

   Solution: If we let (x, y, z) represent a vector whose tip is in the plane, the distance is the projection of $(x, y, z) - (1, 2, 3)$ onto the unit normal $\dfrac{(4, 5, 6)}{\sqrt{77}} = \dfrac{(4, 5, 6)}{\sqrt{77}}$. The length will be $\left|((x, y, z) - (1, 2, 3)) \cdot \dfrac{(4, 5, 6)}{\sqrt{77}}\right| = \dfrac{|4x + 5y + 6z - (1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6)|}{\sqrt{77}} = \dfrac{|7 - 32|}{\sqrt{77}} = \dfrac{25}{\sqrt{77}}$.

   Alternatively, one could find where the line $\mathsf{x} = (1, 2, 3) + (4, 5, 6)t$ intersects the plane and find the distance between that point and $(1, 2, 3)$.

5. An ant walks on the paraboloid $z = x^2+y^2$. Using the usual orientation for the coordinate axis, the ant starts at the origin and is rising at a rate of 4 units per second. Looking from above, it looks as if the ant is spiralling around the z-axis, starting by the x-axis and making a complete revolution every 5 seconds.

   (a) Find a vector parametric equation for the path of the ant.

   Solution: Clearly, $z = 4t$, where t represents time. Since $x^2 + y^2 = z$, we have $x^2 + y^2 = 4t$. If θ is the angle we visualize looking straight down between the x-axis and the line from the origin to the ant, $x = \sqrt{4t}\cos\theta$ and $y = \sqrt{4t}\sin\theta$. Since θ starts at 0 and increases by 2π every 5 seconds, $\theta = \dfrac{2\pi t}{5}$. We thus obtain

   $\mathsf{r} = (2\sqrt{t}\cos(\frac{2\pi t}{5}), 2\sqrt{t}\sin(\frac{2\pi t}{5}), 4t)$.

   (b) Find the velocity of the ant.

   Solution: $\mathsf{v} = \mathsf{r}' = (\frac{1}{\sqrt{t}}\cos(\frac{2\pi t}{5}) - \frac{4\pi\sqrt{t}}{5}\sin(\frac{2\pi t}{5}), \frac{1}{\sqrt{t}}\sin(\frac{2\pi t}{5}) + \frac{4\pi\sqrt{t}}{5}\cos(\frac{2\pi t}{5}), 4)$.

   (c) Find the acceleration of the ant.

   Solution: $\mathsf{a} = \mathsf{v}' = (-\frac{1}{2t\sqrt{t}}\cos(\frac{2\pi t}{5}) - \frac{4\pi}{5\sqrt{t}}\sin(\frac{2\pi t}{5}) - \frac{8\pi^2\sqrt{t}}{25}\cos(\frac{2\pi t}{5}), \frac{1}{2t\sqrt{t}}\sin(\frac{2\pi t}{5}) + \frac{4\pi}{5\sqrt{t}}\cos(\frac{2\pi t}{5}) - \frac{8\pi^2\sqrt{t}}{25}\sin(\frac{2\pi t}{5}), 0)$

   Represent time by t. Each of your conclusions should be in terms of t.

6. Consider the vector function $\mathsf{x} = (t, 3\sin t, 4\cos t)$, $t \geq 0$.

   (a) Sketch the graph.

   (b) Find the length of the portion of the curve for which $\pi \leq t \leq 2\pi$. *Give your answer in terms of a definite integral. You do not need to evaluate the integral but may do so for extra credit.*

   Solution: Since $\dfrac{d\mathsf{x}}{dt} = (1, 3\cos t, -4\sin t)$, the length is $\int_{\pi}^{2\pi} \sqrt{1 + 9\cos^2 t + 16\sin^2 t}\,dt$.

7. Consider the vector function $\mathsf{x} = (t, t^2, t^3)$, $t \geq 1$.
   (a) Sketch the graph.
   (b) Find the unit tangent vector T at $t = 2$.

   Solution: $\mathsf{v} = (1, 2t, 3t^2)$, so $\mathsf{T} = \dfrac{\mathsf{v}}{|\mathsf{v}|} = \dfrac{(1, 2t, 3t^2)}{\sqrt{1 + 4t^2 + 9t^4}}$.

   When $t = 2$, $\mathsf{T} = \dfrac{(1, 4, 12)}{\sqrt{161}}$.

   (c) Find the normal vector N at $t = 2$.

   Solution: $\mathsf{N} = \dfrac{\mathsf{T}'}{|\mathsf{T}'|}$

   $$\mathsf{T}' = \frac{\sqrt{1 + 4t^2 + 9t^4}(0, 2, 6t) - (1, 2t, 3t^2)\frac{8t + 36t^3}{2\sqrt{1+4t^2+9t^4}}}{1 + 4t^2 + 9t^4}.$$

   When $t = 2$, we get $\mathsf{T}' = \dfrac{\sqrt{161}(0, 2, 12) - (1, 4, 12)\frac{152}{\sqrt{161}}}{161} = \dfrac{2(-76, -143, 54)}{161\sqrt{161}}$.

   Thus $\mathsf{N} = \dfrac{(-76, -143, 54)}{|(-76, -143, 54)|} = \dfrac{(-76, -143, 54)}{\sqrt{29141}}$.

   (d) Find the binormal vector B at $t = 2$.

   Solution: $\mathsf{B} = \mathsf{T} \times \mathsf{N} = \dfrac{(1, 4, 12) \times (-76, -143, 54)}{\sqrt{161}\sqrt{29141}}$

   $$(1, 4, 12) \times (-76, -143, 54) = \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ 1 & 4 & 12 \\ -76 & -143 & 54 \end{vmatrix} = (1932, -966, 161)$$

   We thus have $\mathsf{B} = \dfrac{(1932, -966, 161)}{\sqrt{161}\sqrt{29141}}$.

   We can simplify this by observing B is a unit vector in the direction of $(1932, -966, 161) = 161(12, -6, 1)$, so $\mathsf{B} = \dfrac{(12, -6, 1)}{|(12, -6, 1)|} = \dfrac{(12, -6, 1)}{\sqrt{181}}$.

   (e) Verify T, N and B are mutually orthogonal.

   Solution: We calculate each of the dot products. Each comes out to be 0.

   (f) Find the curvature κ at $t = 2$.

   Solution: $\kappa = \dfrac{|\mathsf{T}'|}{\frac{ds}{dt}}$.

   When $t = 2$, $\mathsf{T}' = \dfrac{2(-76, -143, 54)}{161\sqrt{161}}$, so $|\mathsf{T}'| = \dfrac{2\sqrt{29141}}{161\sqrt{161}} = \dfrac{2\sqrt{181}}{161}$.

   Also, when $t = 2$, $\mathsf{v} = (1, 4, 12)$, so $\dfrac{ds}{dt} = |\mathsf{v}| = \sqrt{161}$.

   We thus have $\kappa = \dfrac{\left(\frac{2\sqrt{181}}{161}\right)}{\sqrt{161}} = \dfrac{2\sqrt{181}}{161\sqrt{161}}$.

8. A jogger runs leisurely at a constant pace of a six minute mile around a quarter mile track. The track consists of two parallel straighaways connected by two semicircles, each 110 yards long.

   (a) Set up an appropriate coordinate system.

   Solution: We set the origin at the center of the track. The x-axis is parallel to the straight portion, oriented in the direction the jogger starts out. We orient the y-axis so the y-coordinate is negative when the jogger begins.

   Since each semicircle is of length 110 (measured in yards), we effectively have part of a circle with circumference 220. Letting r be the radius, we have $2\pi r = 220$, so $r = \frac{110}{\pi}$

   We will consider the horizontal portion to be going east-west, with the jogger starting out going to the east, then counterclockwise around a semicircle, then going

(c) Find the acceleration of the runner when he or she is exactly halfway through one of the semicircles. ***Extra Credit: Obtain the acceleration again using an entirely di erent method.***

Solution: Around the first curve, $\mathbf{r} = (55, 0) + \frac{110}{\pi}(\cos(-\frac{\pi}{2} + \frac{8\pi}{3}(t - \frac{3}{8})), \sin(-\frac{\pi}{2} + \frac{8\pi}{3}(t - \frac{3}{8})))$, so $\mathbf{v} = \frac{110}{\pi}(-\sin(-\frac{\pi}{2} + \frac{8\pi}{3}(t - \frac{3}{8})), \cos(-\frac{\pi}{2} + \frac{8\pi}{3}(t - \frac{3}{8}))) \cdot \frac{8\pi}{3}$ and $\mathbf{a} = \frac{110}{\pi}(-\cos(-\frac{\pi}{2} + \frac{8\pi}{3}(t - \frac{3}{8})), -\sin(-\frac{\pi}{2} + \frac{8\pi}{3}(t - \frac{3}{8}))) \cdot (\frac{8\pi}{3})^2$.