Math 210 MW 11:15-12:30, Th 12:30-1:45

13 Vectors and the Geometry of Space

13.1 Three Dimensional Coordinate Systems

Coordinate axes, Right-hand rule
Coordinate planes, octants
Distance formula
Sphere
805/1, 3, 5, 7, 9, 11, 15, 23, 25, 35
805/13, 17, 29, 30, 37

13.2 Vectors

Vector (magnitude, direction), initial point, terminal point
Addition (Parallelogram Law), scalar multiplication, subtraction
Components, Position vector (from origin)
Length, magnitude
Properties: commutative, associative (addition, scalar multiplication), 0, inverse, distributive, multiplication by 1
Standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$
Unit vector
813/1, 3, 4, 7, 13, 17, 19, 23, 27, 31, 35
813/9, 15, 21, 29, 30, 39, 43

13.3 Dot Product

Definition
Properties: $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, commutative, distributive, scalar multiplication, $\mathbf{0} \cdot \mathbf{a} = 0$
$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ (Proof-Law of Cosines)
Orthogonal
Direction angles α, β, γ
Direction cosines
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
Scalar projection $\text{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$
Vector projection $\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$
820/1, 3, 5, 7, 15, 21, 23, 27, 29, 35, 47, 57
820/9, 17, 31, 37, 41, 49, 58

13.4 Cross Product

Definition, mnemonic using determinants
$\mathbf{a} \times \mathbf{b}$ orthogonal to $\mathbf{a}$ and $\mathbf{b}$.
$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ (Proof - $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$)
Length of cross product = area of parallelogram
Properties: anti-commutative, multiplication by scalar, distributive, triple product, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
Volume of parallopiped = scalar triple product
Torque $\tau = \mathbf{r} \times \mathbf{F}$
828/1, 9, 15, 17, 23, 25
828/3, 33, 39, 45

13.5 Equations of Lines and Planes

Lines

Vector Equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$
Parametric Equations $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$
Direction Numbers
Symmetric Equations $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$
Line segment: restrict t
838/1, 3, 7, 15, 19
838/5, 9, 17, 21

Planes

Normal vector
Vector equation: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$
Scalar equation: $a(x - x_0) + b(y - y)_0 + c(z - z_0) = 0$
Parallel, orthogonal planes
Distance from point to plane $\dfrac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|}$
838/23, 25, 39, 47, 53, 55
838/27, 33, 41, 49, 54

13.6 Cylinders and Quadric Surfaces

Definition: Cylinder - lines parallel to a given line passing through a place curve
Quadric surface - graph of second degree equation in three variables
Ellipsoid, elliptic paraboloid, hyperbolic paraboloid, hyperboloid of one sheet, hyperboloid of two sheets
849/1, 3, 5, 9, 11, 13, 21-28, 29
849/15, 31

14 Vector Functions

14.1 Vector Functions and Space Curves

Vector Valued Function $\mathbf{r}(t) =< f(t), g(t), h(t) >= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$
Limit, continuity
Space curves, parametric equations
858/1, 3, 7, 15, 19-24
858/9, 11, 25, 39

14.2 Derivatives and Integrals of Vector Functions

Definition - derivative, integral
Tangent line, unit tangent $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$
Smooth curve: $\mathbf{r}'$ is continuous and non-zero
Properties of derivatives: term-by-term, product and chain rules.
864/1, 3, 9,11, 17, 23, 31, 33
864/5, 8, 15, 19, 27, 35, 37

14.3 Arc Length and Curvature

$L = \int_a^b |\mathbf{r}'(t)|\, dt$
Arc length function $s(t) = \int_a^t |\mathbf{r}'(u)|\, du$
$\dfrac{ds}{dt} = |\mathbf{r}'(t)|$
Parametrize curve with respect to arc length: Solve for t in terms of s
Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$
$\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$
For plane curve $y = f(x)$, $\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$
Normal Vector $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$
Binormal vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$
Normal plane - determined by $\mathbf{N}$ and $\mathbf{B}$
Osculating plane - determined by $\mathbf{T}$ and $\mathbf{N}$
872/1, 3, 13, 37, 39
872/5, 15, 27

14.4 Motion in Space: Velocity and Acceleration

$\mathbf{v}(t) = \mathbf{r}'(t)$
$\mathbf{a}(t) = \mathbf{v}'(t)$
Newton's Second Law of Motion: $\mathbf{F} = m\mathbf{a}$
$\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$
882/1, 3, 5, 9, 19, 23, 31
882/7, 11, 25, 33

15 Partial Derivatives

15.1 Functions of Several Variables

Definition 1 (Function of Two Variables). $f(x, y)$

Independent variables, dependent variable
Domain, range
Graph
Contour or Level Curves - graphs of $f(x, y) = k$
Functions of three or more variables
Three views: function of n real variables $x_1, x_2, \ldots, x_n$, function of a single point variable $(x_1, x_2, \ldots, x_n)$, function of a vector variable $\mathbf{x} = < x_1, x_2, \ldots, x_n >$
901/1, 3, 5, 7, 11, 21
901/9, 15, 25

15.2 Limits and Continuity

Definition $\lim_{(x,y)\to(a,b)} f(x, y) = L$
Definition - Continuity
Functions of three or more variables
913/1, 5, 7, 19, 27, 29, 37
913/9, 11, 31

15.3 Partial Derivatives

Definition $f_x(a,b)$, $f_y(a,b)$, $f_x(x,y)$, $f_y(x,y)$
Notations: f_x, $\frac{\partial f}{\partial x}$, $\frac{\partial z}{\partial x}$, f_1, $D_1 f$, $D_x f$
Functions of more than two variables
Higher Derivatives
Notation: $(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$

Theorem 1 (Clairaut's Theorem). *If f_{xy} and f_{yx} are both continuous on a disk containing (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.*

924/1, 13, 15, 35, 41, 45, 53
924/17, 19, 37, 47, 57

15.4 Tangent Planes and Linear Approximations

Tangent Plane: $z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$ (If partial derivatives are continuous.)
Linear or Tangent Plane Approximation
Definition: f differentiable if $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Theorem 2. *Partials continuous nearby implies function differentiable.*

Total differential: $dz = f_x(x,y)dx + f_y(x,y)dy$
935/1, 3, 7, 11, 17, 23, 29, 31
935/5, 13, 25, 33

15.5 Chain Rule

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$ $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$, $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$
Implicit Differentiation:

$y = f(x)$ defined by $F(x,y) = 0$ implies $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$ implies $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$.

$z = f(x,y)$ defined by $F(x,y,z) = 0 \implies \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$
943/1, 3, 7, 13, 21, 35
943/5, 9, 11, 23, 45

15.6 Directional Derivatives and the Gradient Vector

Definition 2 (Directional Derivative). *Unit vector* $\mathbf{u} = < a, b >$,
$D_u f(x_0, y_0) = \lim_{h \to 0} \dfrac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$.

Theorem 3. $D_u f(x, y) = f_x(x, y)a + f_y(x, y)b$

$D_{\mathbf{u}} f(x, y) = < f_x(x, y), f_y(x, y) > \cdot \mathbf{u}$.

Definition 3 (Gradient). $grad f = \bigtriangledown f = < f_x, f_y >$

Theorem 4. *Maximum value of directional derivative is* $| \bigtriangledown f|$ *and occurs in the direction of* $\bigtriangledown f$.

Tangent plane to level surface $F(x, y, z) = k$: $\bigtriangledown F \cdot < x - x_0, y - y_0, z - z_0 > = 0$
Tangent plane to $z = f(x, y)$: $z - z_0 = f_x(x - x_0) + f_y(y - y_0)$
956/1, 5, 7, 11, 13, 21, 27
956/9, 15, 23, 29

15.7 Maximum and Minimum Values

Definition: local maximum, local minimum, absolute maximum, absolute minimum

Theorem 5. f *has local extremum and partials exist* $\Longrightarrow$ *partials equal 0.*

Critical point (stationary point) - partials are 0 or a partial doesn't exist
Second Derivative Test: Critical point, second partials continuous, $D = f_{xx} f_{yy} - (f_{xy})^2$.

- $D > 0$, $f_{xx} > 0$ implies local minimum
- $D > 0$, $f_{xx} < 0$ implies local maximum
- $D < 0$ implies saddle point

f continuous on closed set implies f has absolute extrema
966/1, 5, 7, 27, 35, 37, 43
966/9, 15, 29, 39, 45, 48

15.8 Lagrange Multipliers

Find extrema for $f(x, y, z)$ subject to constraint $g(x, y, z) = k$.
Solve: $\bigtriangledown f = \lambda \bigtriangledown g$, $g(x, y, z) = k$.
Two constraints $g(x, y, z) = k$, $h(x, y, z) = c$: Solve $\bigtriangledown f = \lambda \bigtriangledown g + \mu \bigtriangledown h$
976/3, 7
976/5, 9

16 Multiple Integrals

16.1 Double Integrals Over Rectangles

Riemann Sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$
Definition: $\int\int_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$
Application: Volume
Numerical calculation: Midpoint Rule
Properties: $\int\int_R f(x,y) \pm g(x,y)\,dA$, $\int\int_R kf(x,y)\,dA$
$f(x,y) \geq g(x,y) \implies \int\int_R f(x,y)\,dA \geq \int\int_R g(x,y)\,dA$
994/1, 3, 11
994/13, 14

16.2 Iterated Integrals

Iterated Integral

Theorem 6 (Fubini's Theorem). *If f is continuous on a rectangle R,* $\int\int_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$.

1000/1, 3, 11, 13
1000/5, 9, 15

16.3 Double Integrals over General Regions

If region D lies in a rectangle R, define $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in R \\ 0 & \text{if } (x,y) \notin R. \end{cases}$
Type I Region: Between two functions, vertical sides - convert to iterated integral
Type II Region: Horizontal sides
Properties: Sum or difference, multiplication by constant, $f(x,y) \geq g(x,y)$, integral over union of non-overlapping regions, $m \leq f(x,y) \leq M$
1008/1, 7, 9, 19
1008/3, 11, 13, 37

16.4 Double Integrals in Polar Coordinates

$\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$
1014/1-3, 7, 9, 17, 21, 29
1014/4-6, 11, 15, 19, 23, 31, 33

16.5 Applications of Double Integrals

Density and mass
Moments and center of mass
Moment of inertia
1024/3, 5, 11
1024/7, 15

16.6 Triple Integrals

Definition
Turn into iterated integral
Applicatons:
Volume = $\iiint_E dV$
Mass, center of mass, moments, centroid, moment of inertia
1035/1, 3, 7, 9, 17
1035/5, 11, 25, 27

16.7 Triple Integrals in Cylindrical Coordinates

$dV \to r\,dz\,dr\,d\theta$
1040/1,3, 7,9, 15, 23
1040/5, 17, 19, 25, 27

16.8 Triple Integrals in Spherical Coordinates

$dV \to \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$
1046/1,3,6, 7,9,11, 15, 21
1046/5,13, 17, 19, 33, 39

16.9 Change of Variables in Multiple Integrals

Transformation $T(u,v) = (x,y)$
$x = g(u,v)$, $y = h(u,v)$ or $x = x(u,v)$, $y = y(u,v)$
Let S be rectangle with sides $(\Delta u, \Delta v)$. Image $R = T(S)$ is approximately a rectangle with sides $< \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} > \Delta u$, $< \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} > \Delta v$.

Area is approximately the Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v$.

So $\Delta A \approx \frac{\partial(x,y)}{\partial(u,v)} \Delta u \Delta v$.

$\iint_R f(x,y)\, dA = \iint_S f(g(u,v), h(u,v)) \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| du\, dv$.

Example: Change to polar coordinates.
Triple Integrals
1057/1, 7, 11, 15, 17
1057/3, 5, 9, 13, 23, 35

17 Vector Calculus

17.1 Vector Fields

Definition 4 (Vector Field). *Vector function* $\mathbf{F}$ *assigning* $(x,y) \to \mathbf{F}(x,y)$.

Examples: velocity field, gravitational field, force field, gradient vector field

Definition 5 (Conservative Vector Field). $\mathbf{F}$ *is conservative if* $\mathbf{F} = \bigtriangledown f$ *for some potential function* f.

1068/1, 11-14, 21
1068/3, 15-18, 25

17.2 Line Integrals

Definition: $\int_C f(x,y)\,ds$ in terms of Riemann Sum.

Calculation: $ds = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}$

Example: Mass of wire, center of mass

Variations: $\int_C f(x,y)\,dx$, $\int_C f(x,y)\,dy$, $\int_C P(x,y)\,dx + Q(x,y)\,dy$

Line integrals in space

Line integrals of vector fields: Work = $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$

Definition 6 (Line Integral of $\mathbf{F}$ along C). $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_C \mathbf{F} \cdot \mathbf{T}\,ds.$

1079/1, 3, 9, 17, 19, 31
1079/5, 7, 11, 21, 39

17.3 Fundamental Theorem for Line Integrals

Theorem 7. $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$

Definition: Independence of path

Independent of path if and only if integral along any closed path is 0.

Theorem: $\int_C \mathbf{F} \cdot d\mathbf{r}$ independent of path $\Longrightarrow$ $\mathbf{F}$ is a conservative vector field.

Theorem: $\mathbf{F} = < P, Q >$ conservative $\Longrightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Simple curve, simply connected

Theorem: $\mathbf{F} = < P, Q >$ on open simply-connected region and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\Longrightarrow$ $\mathbf{F}$ is conservative.

1089/1, 3, 5, 13, 15, 19
1089/7, 17, 21, 33

17.4 Green's Theorem

Theorem 8 (Green's Theorem). $\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

1096/1, 7, 13
1096/3, 9, 15

17.5 Curl and Divergence

curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

Theorem 9. *If f has continuous second-order partial derivatives,* $\nabla \times (\nabla f) = \mathbf{0}$

Corollary: If $\mathbf{F}$ is conservative, then $\nabla \times \mathbf{F} = \mathbf{0}$.

Definition 7 (Divergence). $\mathit{div}\, \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Theorem 10. $\nabla \cdot \nabla \times \mathbf{F} = 0$

Vector Form of Green's Theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$.
$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_D \nabla \cdot \mathbf{F}\, dA$
1104/1, 3, 13, 23
1104/5, 7, 31

17.6 Parametric Surfaces and their Areas

Parametric Surface $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$
Surface of revolution - from $y = f(x)$: $x = x$, $y = f(x)\cos\theta$, $z = f(x)\sin\theta$.
Tangent planes: use tangent vectors $< \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} >, < \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} >$
Surface area - of surface given by $\mathbf{r}(u, v)$
Area is $\iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$
1114/1, 11-16, 19, 35
1114/3, 23, 39

17.7 Surface Integrals

$\iint_S f(x, y, z)\, dS$ as a Riemann Sum

For surface $z = g(x, y)$, $\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{\frac{\partial z}{\partial x}^2 + \frac{\partial z}{\partial y}^2 + 1}\, dA$

In general, $dS \to |\mathbf{r}_u \times \mathbf{r}_v|\, dA$
Surface integral of vector field $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$. Called flux of $\mathbf{F}$ across S.
1127/5, 7, 19
1127/9, 11, 21

17.8 Stokes' Theorem

Theorem 11 (Stokes' Theorem). $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$.

1133/3, 7, 13
1133/5, 9, 17

17.9 Divergence Theorem

Theorem 12 (Divergence Theorem). $\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F}\, dV$.

1139/1, 3, 7, 22
1139/5, 9, 23