Coordinate Systems and Parametrizations

One can generate parametric equations for certain curves, surfaces and even solids by looking at equations for certain figures in different coordinate systems along with the conversions between those coordinate systems and the Cartesian Coordinate System.

Circles

In polar coordinates, the equation of the unit circle with center at the origin is $r = 1$.

Suppose we take the formulas

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

and replace r by 1. We get

$$\begin{aligned} x &= \cos\theta \\ y &= \sin\theta. \end{aligned}$$

If we let θ go between 0 and 2π, we will trace out the unit circle, so we have the parametric equations

$$\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \\ 0 \leq &\ \theta \leq 2\pi \end{aligned}$$

for the circle.

If we want, we can replace θ by t to get

$$\begin{aligned} x &= \cos t \\ y &= \sin t \\ 0 \leq &\ t \leq 2\pi. \end{aligned}$$

If we restrict t to some other interval, we can get a part of the circle.

With slight variations, we can get other circles:

$$\begin{aligned} x &= a\cos t \\ y &= a\sin t \\ 0 \leq &\ t \leq 2\pi \end{aligned}$$

or ellipses

$$\begin{aligned} x &= a\cos t \\ y &= b\sin t \\ 0 \le \quad & t \le 2\pi. \end{aligned}$$

The Unit Disk

In polar coordinates, the inequality $r \le 1$ gives the unit disk.

If we take the formulas for rectangular coordinates in terms of polar coordinates,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned}$$

and restrict $r \le 1$, we get parametric equations for the unit disk

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ 0 \le r \le 1, &\quad 0 \le \theta \le 2\pi. \end{aligned}$$

We can disguise the origins by replacing r and θ by u and v to get

$$\begin{aligned} x &= u\cos v \\ y &= u\sin v \\ 0 \le u \le 1, &\quad 0 \le v \le 2\pi. \end{aligned}$$

Lines

In polar coordinates, the equation $\theta = \alpha$, for some $\alpha \in \mathbb{R}$, yields a line through the origin.

If we take

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

and let $\theta = \alpha$, and let $a = \cos\alpha$, $b = \sin\alpha$,

we get

$$\begin{aligned} x &= ar \\ y &= br. \end{aligned}$$

As we let r vary, we get points on a line. We can even eliminate r to get $\dfrac{y}{b} = \dfrac{x}{a}$ or $y = \frac{b}{a}x$.

Cylinders

In Cylindrical Coordinates, the equation $r = 1$ gives a cylinder of radius 1.

If we take

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

and replace r by 1, we get

$$\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \\ z &= z. \end{aligned}$$

If we restrict θ and z, we get parametric equations for a cylinder of radius 1.

$$\begin{gathered} x = \cos\theta \\ y = \sin\theta \\ z = z \\ 0 \le \theta \le 2\pi, \quad 0 \le z \le 4 \end{gathered}$$

gives a cylinder of radius 1 and height 4.

More generally,

$$\begin{gathered} x = r\cos\theta \\ y = r\sin\theta \\ z = z \\ 0 \le \theta \le 2\pi, \quad 0 \le z \le h \end{gathered}$$

gives a cylinder of radius r and height h.

If we want, we can change the names of the parameters:

$$\begin{aligned} x &= r\cos u \\ y &= r\sin u \\ z &= v \\ 0 \le u \le 2\pi,& \quad 0 \le v \le h \end{aligned}$$

gives the same cylinder of radius r and height h.

Planes

In Cylindrical Coordinates, the equation $\theta = \alpha$ gives a plane which contains the z axis and which is perpendicular to the xy plane.

If we take the conversion formulas

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

and let $\theta = \alpha$, $a = \cos\alpha$, $b = \sin\alpha$, we get

$$\begin{aligned} x &= ar \\ y &= br \\ z &= z. \end{aligned}$$

These are parametric equations of a plane.

Spheres

In Spherical Coordinates, the equation $\rho = 1$ gives a unit sphere.

If we take the conversion formulas

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

and replace ϕ by 1, we get

$$\begin{aligned} x &= \sin\phi\cos\theta \\ y &= \sin\phi\sin\theta \\ z &= \cos\phi. \end{aligned}$$

Putting appropriate ranges for ϕ and θ,

$$
\begin{aligned}
x &= \sin\phi\cos\theta \\
y &= \sin\phi\sin\theta \\
z &= \cos\phi \\
0 \le \phi \le \pi, &\quad 0 \le \theta \le 2\pi
\end{aligned}
$$

gives parametric equations for the unit sphere.

Changing the Parameters

We can disguise the origins of the parametrization by replace ϕ by u and θ by v:

$$
\begin{aligned}
x &= \sin u\cos v \\
y &= \sin u\sin v \\
z &= \cos u \\
0 \le u \le \pi, &\quad 0 \le v \le 2\pi
\end{aligned}
$$

Variations

$$
\begin{aligned}
x &= r\sin u\cos v \\
y &= r\sin u\sin v \\
z &= r\cos u \\
0 \le u \le \pi, &\quad 0 \le v \le 2\pi
\end{aligned}
$$

will give a sphere of radius r.

$$
\begin{aligned}
x &= a\sin u\cos v \\
y &= b\sin u\sin v \\
z &= c\cos u \\
0 \le u \le \pi, &\quad 0 \le v \le 2\pi
\end{aligned}
$$

will give an ellipsoid.

More Variations

Restricting the domains of the parameters gives us part of the sphere.

$$
\begin{aligned}
x &= \sin u\cos v \\
y &= \sin u\sin v \\
z &= \cos u \\
0 \le u \le \pi/2, &\quad 0 \le v \le 2\pi
\end{aligned}
$$

gives the top half of the sphere.

$$\begin{aligned} x &= \sin u \cos v \\ y &= \sin u \sin v \\ z &= \cos u \\ 0 \le u \le \pi, &\quad 0 \le v \le \pi \end{aligned}$$

gives the half sphere for which $y \geq 0$.

$$\begin{aligned} x &= \sin u \cos v \\ y &= \sin u \sin v \\ z &= \cos u \\ 0 \le u/2 \le \pi, &\quad 0 \le v \le \pi/2 \end{aligned}$$

gives an eighth of the sphere.