Vector Functions

A vector function is simply a function whose codomain is $\mathbb{R}^n$. In other words, rather than taking on real values, it takes on vector values.

We will often use the notation $\mathbf{x} = \mathbf{x}(t)$ to denote a vector function. *Note: The text generally uses* $\mathbf{r}(t)$ *rather than* $\mathbf{x}(t)$*, but your instructor is used to the former.*

Suppose $\mathbf{x} : \mathbb{R} \to \mathbb{R}^3$. We may write $\mathbf{x}(t) =< f(t), g(t), h(t) >$ or $\mathbf{x}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$.

The functions $f(t)$, $g(t)$, $h(t)$ are referred to as the component functions. The independent variable t is also referred to as the parameter.

Limits, Continuity, Differentiation and Integration

The definitions of limits, continuity, derivatives and integrals of vector functions are straighforward generalizations of the corresponding definitions for ordinary functions.

In practice, we deal with these calculations and the concept of continuity through the component functions.

Limits

Recall:

Definition 1 (Limit). $\lim_{x\to x_0} f(x) = L$ *if for every* $\epsilon > 0$ *there is some* $\delta > 0$ *such that* $|f(x) - L| < \epsilon$ *whenever* $0 < |x - x_0| < \delta$.

We can get a definition for the limit of a vector function by changing the function to a vector function and the limit to a vector. We'll simply take the definition above, replace x by t, f by $\mathbf{x}$, x by t and L by $\mathbf{x}_0$.

Definition 2 (Limit). $\lim_{t\to t_0} \mathbf{x}(t) = \mathbf{x}_0$ *if for every* $\epsilon > 0$ *there is some* $\delta > 0$ *such that* $|\mathbf{x}(t) - \mathbf{x}_0| < \epsilon$ *whenever* $0 < |t - t_0| < \delta$.

Effectively, $\lim_{t\to t_0} < f(t), g(t) > = < \lim_{t\to t_0} f(t), \lim_{t\to t_0} g(t) >$.

Continuity

For ordinary functions:

Definition 3 (Continuity). *A function* f *is continuous at* x_0 *if* $\lim_{x\to x_0} f(x) = f(x_0)$.

For vector functions:

Definition 4 (Continuity). *A vector function* $\mathbf{f}$ *is continuous at* x_0 *if* $\lim_{x\to x_0} \mathbf{f}(x) = \mathbf{f}(x_0)$.

Or, using our usual notation:

Definition 5 (Continuity). *A vector function* $\mathbf{x}$ *is continuous at* t_0 *if* $\lim_{t\to t_0} \mathbf{x}(t) = \mathbf{x}(t_0)$.

Derivatives

Recall the definition of a derivative of an ordinary function:

Definition 6 (Derivative). $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ *wherever the limit exists.*

For vector functions,

Definition 7 (Derivative). $\mathbf{f}'(x) = \lim_{h\to 0} \frac{\mathbf{f}(x+h)-\mathbf{f}(x)}{h}$ *wherever the limit exists.*

Or, using our usual notation,

Definition 8 (Derivative). $\mathbf{x}'(t) = \lim_{h\to 0} \frac{\mathbf{x}(t+h)-\mathbf{x}(t)}{h}$ *wherever the limit exists.*

In practice, if $\mathbf{x}(t) =< f(t), g(t) >$, then $\mathbf{x}'(t) =< f'(t), g'(t) >$.

Space Curves

The graph of a vector function $\mathbf{x}(t)$ is the set of tips of the vectors $\mathbf{x}(t)$ when the initial point is placed at the origin and t ranges over the domain.

If $\mathbf{x}'(t)$ is continuous and non-zero, we say the curve is a smooth curve.

If $\mathbf{x}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then the same curve may be described by the scalar parametric equations:

$$\begin{aligned} x &= f(t) \\ y &= g(t) \\ z &= h(t) \end{aligned}$$

Velocity, Speed and Arc Length

Suppose $\mathbf{x}(t)$ represents the position of a particle at time t. $\mathbf{x}(t)$ traces out a space curve. Since the derivative represents rate of change, the velocity $\mathbf{v} = \mathbf{v}(t)$ of the particle will equal the derivative $\mathbf{x}'(t)$.

The speed of the particle is equal to the magnitude $|\mathbf{v}|$ of the velocity. We often denote the speed by $\dfrac{ds}{dt}$.

Since distance may be obtained by integrating speed, the distance travelled by the particle as t goes from t_1 to t_2 would equal $\int_{t_1}^{t_2} |\mathbf{v}(t)|\, dt$.

The distance traversed is the same as the length of a portion of the curve. If we denote the arc length by s, we get $s = \int_{t_1}^{t_2} \frac{ds}{dt}\, dt$.

Back to the Past

Consider an ordinary curve $y = f(x)$, $a \leq x \leq b$. We can parametrize it, using the *canonical parametrization*, as $\mathbf{x}(t) = < t, f(t) >$, for $a \leq t \leq b$. Then $\mathbf{v} = < 1, f'(t) >$, so $\frac{ds}{dt} = \sqrt{1 + f'(t)^2}$, so we get $s = \int_a^b \sqrt{1 + f'(t)^2}\, dt$.

This should look familiar; it's the old formula for arc length.

Parametrization With Respect to Arc Length

We may look at the arc length s as a function $s = \alpha(t)$, where $\alpha(t)$ is the length of the portion of the curve between a fixed initial point and t.

Since s obviously increases as t increases, $\alpha(t)$ is one-to-one and hence has an inverse.

Thus, we may write $t = \alpha^{-1}(s)$, so $\mathbf{x}(t) = \mathbf{x}(\alpha^{-1}(s))$ may be thought of a parametrization of the same curve with respect to arc length.

We rarely actually write out the parametrization, but it still comes in handy.

We also always have the relationship $|\mathbf{v}| = \frac{ds}{dt}$.

Unit Tangent

$\mathbf{v}$ is tangent to the curve. *Try to convince yourself of this. We won't prove it. In fact, try to come up with a definition of tangency when dealing in three or more dimensions!*

Definition 9 (Unit Tangent). $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$ *is called the unit tangent.*

Obvious Properties:

- $|\mathbf{T}| = 1$
- $\mathbf{v} = \frac{ds}{dt}\mathbf{T}$ In other words, the velocity vector is in the direction of the unit tangent and its length is the speed.

Theorem 1 (Claim). $\mathbf{T}' \perp \mathbf{T}$

Proof. Since $|\mathbf{T}| = 1$, $\mathbf{T} \cdot \mathbf{T} = 1$. Di�erentiating, $\frac{d}{dt}(\mathbf{T} \cdot \mathbf{T}) = \frac{d}{dt}(1)$. $2\mathbf{T} \cdot \mathbf{T}' = 0$. $\mathbf{T} \cdot \mathbf{T}' = 0$. □

Definition 10 (Unit Normal). $N = \dfrac{\mathbf{T}'}{|\mathbf{T}'|}$

Thus $\mathbf{N}$ is a unit vector orthogonal to $\mathbf{T}$.

Definition 11 (Binormal). $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

The binormal is a unit vector orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.

Curvature

Since the unit tangent doesn't change its length, any change has to do with how fast the curve is curving. We thus use the rate at which the unit tangent changes to measure *curvature*. In order to make it independent of the parametrization of the curve, we define curvature to be the magnitude of the rate at which the unit tangent changes when the parametrization is with respect to arc length.

Definition 12 (Curvature). $\kappa = \left|\dfrac{dT}{ds}\right|$

Using the Chain Rule, $\dfrac{d\mathbf{T}}{dt} = \dfrac{d\mathbf{T}}{ds}\dfrac{ds}{dt}$, so $\dfrac{d\mathbf{T}}{ds} = \dfrac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}}$, so $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{\left|\frac{d\mathbf{T}}{dt}\right|}{\frac{ds}{dt}}$.

Acceleration

Definition 13 (Acceleration). $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$

Since $\mathbf{v} = \dfrac{ds}{dt}\mathbf{T}$, we get $\mathbf{a} = \dfrac{d^2s}{dt^2}\mathbf{T} + \dfrac{ds}{dt}\dfrac{d\mathbf{T}}{dt}$.

Since $\dfrac{d\mathbf{T}}{dt} = \left|\dfrac{d\mathbf{T}}{dt}\right|\mathbf{N} = \kappa\dfrac{ds}{dt}\mathbf{N}$, we get

$$\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.$$

This tells us that the tangential component of acceleration is equal to the rate at which the speed is changing, while the normal component of acceleration is jointly proportional to the curvature and the square of the speed. Recall that kinetic energy is also proportional to the square of the speed.