Three Dimensional Euclidean Space

We set up a coordinate system in space (three dimensional Euclidean space) by adding third axis perpendicular to the two axes in the plane (two dimensional Euclidean space).

Usually the axes are called $x$, $y$ and $z$, but that isn’t essential. The three axes form a right hand system, in the sense that if one uses a screwdriver on a screw, turning clockwise from the $x$-axis towards the $y$-axis, the screw moves in the direction of the $z$-axis.

Coordinates of a Point

In two dimensions, the $x$-coordinate represents a signed distance in the direction of the positive or negative $x$-axis and the $y$-coordinate represents a signed distance in the direction of the positive or negative $y$-axis.

In three dimensions, the $x$-coordinate represents a signed distance in the direction of the positive or negative $x$-axis, the $y$-coordinate represents a signed distance in the direction of the positive or negative $y$-axis and $z$-coordinate represents a signed distance in the direction of the positive or negative $z$-axis.

Drawing the Coordinate Axes

To do the impossible and draw the three perpendicular axes in a plane, we draw the $y$-axis going horizontally to the right, the $z$-axis vertically going up, and the $x$-axis making an angle of $\frac{3\pi}{8}$ or $135^\circ$ with the other two axes. We visualize the $x$ and $y$-axes as being in the horizontal plane and the $z$-axis as being vertical.

Physicists and engineers sometimes draw the $x$ and $y$-axes where they’re drawn for $\mathbb{R}^2$ and the $z$-axis where we draw the $x$-axis.

Distance Between Two Points

Consider points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$. Let $P_3$ be the point with coordinates $(x_2, y_2, z_1)$. $|P_1P_3|$ is clearly the same as the distance between the points $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ in the $xy$-plane, so by the distance formula in $\mathbb{R}^2$, $|P_1P_3|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

Since $P_2P_3$ is a vertical line segment, $|P_2P_3| = |z_2 - z_1|$.

Since $P_1P_3$ and $P_3P_2$ form the legs of a right triangle with hypotenuse $P_1P_2$, we may use the Pythagorean Theorem to get $|P_1P_3|^2 + |P_3P_2|^2 = |P_1P_2|^2$, so

$$[(x_2 - x_1)^2 + (y_2 - y_1)^2] + (z_2 - z_1)^2 = |P_1P_2|^2.$$

We thus get the natural generalization of the distance formula to three dimensions:
The distance \( s \) between points with coordinates \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) is given by

\[
s^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = |P_1P_2|^2.
\]

Equations of Spheres

Since a sphere with center \((x_0, y_0, z_0)\) and radius \( r \) consists of all points \((x, y, z)\) a distance \( r \) from \((x_0, y_0, z_0)\), the distance formula immediately shows its equation is

\[
(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.
\]

Example: The sphere with center \((2,5,-3)\) and radius 7 has equation \((x - 2)^2 + (y - 5)^2 + (z + 3)^2 = 49\). Remember, \( z - (-3) = z + 3 \).

Example: \((x + 4)^2 + (y - 2)^2 + (z - 8)^2 = 43\) is an equation for the sphere with center \((-4, 2, 8)\) and radius \(\sqrt{43}\).

Completing the Square

If an equation has all three variables occurring to the second degree, with coefficient 1, but also has some or all occurring to degree one, the method of completing the square can be used to put it in the standard form for an equation of a sphere.

Example: \( x^2 + 6x + y^2 - 8y + z^2 + 14z = 7 \).
\[
(x + 3)^2 = x^2 + 6x + 9, \text{ so } x^2 + 6x = (x + 3)^2 - 9
\]
\[
(y - 4)^2 = y^2 - 8y + 16, \text{ so } y^2 - 8y = (y - 4)^2 - 16
\]
\[
(z + 7)^2 = z^2 + 14z + 49, \text{ so } z^2 + 14z = (z + 7)^2 - 49
\]

Thus, \( x^2 + 6x + y^2 - 8y + z^2 + 14z = 7 \) may be written in the form
\[
[(x + 3)^2 - 9] + [(y - 4)^2 - 16] + [(z + 7)^2 - 49] = 7, \text{ or } (x + 3)^2 + (y - 4)^2 + (z + 7)^2 = 81.
\]

So the equation is for a sphere with center \((-3, 4, -7)\) and radius 9.

Vectors

For a physicist, a vector has magnitude and direction.

For a mathematician, a vector space is a collection of objects satisfying certain conditions and the elements are vectors.

In this course, we will be less abstract. A vector in \(\mathbb{R}^n\), \(n\)-dimensional Euclidean space, will be an \(n\)-tuple.

In \(\mathbb{R}^2\), a vector will be an ordered pair \(<a, b>\) of real numbers.

In \(\mathbb{R}^3\), a vector will be an ordered triple \(<a, b, c>\) or real numbers.

Most of our early examples will be in \(\mathbb{R}^2\), but will easily generalize to \(\mathbb{R}^3\) or higher dimensional Euclidean space.
Geometric Interpretation and Notation

We can visualize the vector \(<a, b>\) as the directed line segment from the origin to the point \((a, b)\), or as any other directed line segment with the same length going in the same direction.

Notation: Vectors are usually printed in boldface, such as \(\mathbf{v} = <a, b>\). It's hard to print in boldface, so when writing vectors by hand one generally puts an arrow above it, such as \(\overrightarrow{v} = <a, b>\).

Addition of Vectors

**Definition 1** (Vector Addition). \(<a, b> + <c, d> = <a + c, b + d>\).

This probably isn’t much of a surprise. *This definition is for \(\mathbb{R}^2\). The generalization to other dimensions should be obvious.*

Geometrically, one may visualize \(\mathbf{v} + \mathbf{w}\) by placing the initial point of \(\mathbf{w}\) at the endpoint of \(\mathbf{v}\). \(\mathbf{v} + \mathbf{w}\) goes from the initial point of \(\mathbf{v}\) to the endpoint of \(\mathbf{w}\).

**Commutativity**

Addition is *commutative*, \(\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}\).

**The Zero Vector**

The vector \(\mathbf{0} = <0, 0>\) is called the *zero vector*.

It satisfies the property \(\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}\) for any vector \(\mathbf{v}\).

**Additive Inverse**

Every vector \(\mathbf{v}\) has an additive inverse, denoted by \(-\mathbf{v}\), such that \(\mathbf{v} + (-\mathbf{v}) = \mathbf{0}\).

It is easy to see \(-<a, b> = <-a, -b>\).

Subtraction of Vectors

**Definition 2** (Vector Subtraction). \(\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})\)

It is easy to see \(<a, b> - <c, d> = <a - c, b - d>\).

This probably isn’t much of a surprise. *This definition is for \(\mathbb{R}^2\). The generalization to other dimensions should be obvious.*

Geometrically, one may visualize \(\mathbf{v} - \mathbf{w}\) as going from the endpoint of \(\mathbf{w}\) to the endpoint of \(\mathbf{v}\).

Subtraction is *not* commutative!

**Scalar Multiplication**

Real numbers are referred to as *scalars*. Multiplication of a vector by a scalar is referred to as scalar multiplication.
Definition 3 (Scalar Multiplication). \( k < a, b > = < ka, kb > \)

Geometrically, if \( k > 0 \), \( k \mathbf{v} \) is a vector in the same direction as \( \mathbf{v} \) with a magnitude \( k \) times as great.

If \( k < 0 \), the \( k \mathbf{v} \) is in the opposite direction from \( \mathbf{v} \).

It is easy to see \( 0 \mathbf{v} = \mathbf{0} \) and \( 1 \mathbf{v} = \mathbf{v} \).

The Distributive Law

Scalar multiplication is distributive under any reasonable interpretation.

For example,

\[
 k(\mathbf{v} + \mathbf{w}) = k\mathbf{v} + k\mathbf{w} \\
(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}
\]

Magnitude of a Vector

Definition 4 (Magnitude or Length). \( | < a, b > | = \sqrt{a^2 + b^2} \)

A vector of length 1 is called a unit vector.

We may find a unit vector in the same direction as a vector \( \mathbf{v} \) by dividing by its length. In other words, we take \( \frac{\mathbf{v}}{|\mathbf{v}|} \).

We haven’t defined scalar division; what we mean is \( \frac{1}{|\mathbf{v}|} \cdot \mathbf{v} \).

Standard Basis Vectors

The unit vectors in the directions of the coordinate axes are called the standard basis vectors and denoted by \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \).

In \( \mathbb{R}^2 \), \( \mathbf{i} = < 1, 0 > \), \( \mathbf{j} = < 0, 1 > \).

In \( \mathbb{R}^3 \), \( \mathbf{i} = < 1, 0, 0 > \), \( \mathbf{j} = < 0, 1, 0 > \), \( \mathbf{k} = < 0, 0, 1 > \).

Any vector can easily be written in terms of the standard basis vectors: \( < a, b, c > = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \).

Dot Product

Definition 5 (Dot Product). \( < a, b, c > \cdot < d, e, f > = ad + be + cf \)

Properties:

- \( \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \)
- \( 0 \cdot \mathbf{v} = 0 \)
- The dot product is commutative: \( \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} \).
- The dot product is distributive over addition \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \).
• $k(v \cdot w) = (kv) \cdot w = v \cdot (kw)$
• $v \cdot w = |v||w| \cos \theta$, where $\theta$ is the angle between the vectors.

Law of Cosines

The formula $v \cdot w = |v||w| \cos \theta$ may be proven using the Law of Cosines.

If we place the initial points of $v$ and $w$ together, then $v, w$ and $v - w$ form a triangle.

Using the Law of Cosines and remembering $v \cdot v = |v|^2$, we have $(v - w) \cdot (v - w) = v \cdot v + w \cdot w - 2|v||w| \cos \theta$.

Multiplying out the dot product on the left, we get

$$v \cdot v - 2v \cdot w + w \cdot w = v \cdot v + w \cdot w - 2|v||w| \cos \theta.$$ 

$$-2v \cdot w = -2|v||w| \cos \theta$$

$$v \cdot w = |v||w| \cos \theta$$

Direction Angles and Direction Cosines

The angles a vector makes with the three coordinate axes are called direction angles and denoted by $\alpha$, $\beta$ and $\gamma$.

The cosines of the direction angles are called the direction cosines, $\cos \alpha$, $\cos \beta$ and $\cos \gamma$.

We know $v \cdot i = |v||i| \cos \alpha$.

If $v = \langle a, b, c \rangle$, since $i = \langle 1, 0, 0 \rangle$ and $|i| = 1$, we get $a = \sqrt{a^2 + b^2 + c^2} \cos \alpha$, so

$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}}.$$ 

Similarly,

$$\cos \beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}}.$$ 

$$\cos \gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$ 

Projections

Definition 6 (Scalar Projection of $v$ on $w$). $\text{comp}_w v = \frac{v \cdot w}{|w|}$

If the angle between the vectors is acute, the scalar projection is the length of the leg along $w$ of the right triangle formed by drawing a line from the tip of $v$ perpendicular to $w$.

If the angle is obtuse, the scalar projection is the negative of the length.

Definition 7 (Vector Projection of $v$ on $w$). $\text{proj}_w v = \left( \frac{v \cdot w}{|w|^2} \right) \frac{w}{|w|} = \frac{v \cdot w}{|w|^2} w$
Geometrically, this is the vector along $w$ whose length is equal to the length of the scalar projection.

**Cross Product**

The cross product $v \times w$ is a vector of length $|v||w|\sin \theta$, where $\theta$ is the angle between $v$ and $w$, orthogonal to both $v$ and $w$, such that $v$, $w$, $v \times w$ form a right-hand triple.

We will come up with a definition and then show it has all the above properties.

If the cross product has the properties indicated above, it follows that:

$i \times j = k$, $j \times k = i$, $k \times i = j$, $j \times i = -k$, $k \times j = -i$, $i \times k = -j$, $i \times i = j \times j = k \times k = 0$.

If the usual rules of algebra, such as the associative and distributive laws, hold for the cross product, we could calculate the cross product of any two vectors by writing them in terms of the standard basis vectors.

**Cross Product**

Letting $v = < x_1, y_1, z_1 >$, $w = < x_2, y_2, z_2 >$, we get

$v \times w = (x_1i + y_1j + z_1k) \times (x_2i + y_2j + z_2k) = x_1x_2i \times i + x_1y_2i \times j + x_1z_2i \times k + y_1x_2j \times i + y_1y_2j \times j + y_1z_2j \times k + z_1x_2k \times i + z_1y_2k \times j + z_1z_2k \times k = 0 + x_1y_2k - x_1z_2j - y_1x_2k + 0 + y_1z_2i + z_1x_2j - z_1y_2i + 0 = (y_1z_2 - y_2z_1)i + (z_1x_2 - z_2x_1)j + (x_1y_2 - x_2y_1)k$.

**Definition 8** (Cross Product). $< x_1, y_1, z_1 > \times < x_2, y_2, z_2 > = < y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1 >$

This is a complicated definition. Fortunately, there’s a convenient mnemonic device involving symbolic determinants that may be used to calculate cross products.

**Determinants**

A matrix is a rectangular array of elements. A square matrix has the same number of rows as columns.

There is a general definition of a determinant of a square matrix. The special case of the determinant of a $3 \times 3$ matrix, with 3 rows and 3 columns, suffices for our purposes.

$$
\begin{vmatrix}
  x_{11} & x_{12} & x_{13} \\
  x_{21} & x_{22} & x_{23} \\
  x_{31} & x_{32} & x_{33}
\end{vmatrix}
= x_{11}x_{22}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} - x_{13}x_{22}x_{31}.
$$
We may think of this as adding the products of elements in each diagonal going down as we go from left to right and subtracting the products of elements in each diagonal going down as we go from right to left.

**Cross Product as a Symbolic Determinant**

Symbolically, \(<a, b, c> \times <d, e, f> = \det \begin{pmatrix} i & j & k \\ a & b & c \\ d & e & f \end{pmatrix} \).

**Properties of the Cross Product**

One can see \(\mathbf{v} \times \mathbf{w}\) is orthogonal to both \(\mathbf{v}\) and \(\mathbf{w}\) by calculating the dot products \(\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})\) and \(\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w})\).

For example, if \(\mathbf{v} = <x_1, y_1, z_1>\) and \(\mathbf{w} = <x_2, y_2, z_2>\), then \(\mathbf{v} \times \mathbf{w} = <y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1>\), so \(\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = x_1(y_1z_2 - y_2z_1) + y_1(x_2z_1 - x_1z_2) + z_1(x_1y_2 - x_2y_1) = x_1y_1z_2 - x_1y_2z_1 + x_2y_1z_2 - x_1y_2z_1 = 0.\)

A similar calculation works for \(\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w})\).

**Magnitude of the Cross Product**

Again, let \(\mathbf{v} = <x_1, y_1, z_1>\) and \(\mathbf{w} = <x_2, y_2, z_2>\), so \(\mathbf{v} \times \mathbf{w} = <y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1>\) and \(|\mathbf{v} \times \mathbf{w}|^2 = (y_1z_2 - y_2z_1)^2 + (x_2z_1 - x_1z_2)^2 + (x_1y_2 - x_2y_1)^2 = (y_1^2z_2^2 - 2y_1y_2z_1z_2 + y_2^2z_1^2) + (x_2^2z_1^2 - 2x_1x_2z_1z_2 + x_1^2z_2^2) + (x_1y_2^2 - 2x_1y_1y_2 + x_2^2y_1^2)\).

Noticing the products of squares suggests looking at the product of \(|\mathbf{v}|^2|\mathbf{w}|^2\).

\[|\mathbf{v}|^2|\mathbf{w}|^2 = (x_1^2 + y_1^2 + z_1^2) + (x_2^2 + y_2^2 + z_2^2) = x_1^2x_2^2 + x_1y_2^2 + x_1^2z_2^2 + y_1^2x_2^2 + y_1^2y_2^2 + y_1^2z_2^2 + z_1^2x_2^2 + z_1^2y_2^2 + z_1^2z_2^2.\]

If one looks at the difference, one gets

\[|\mathbf{v}|^2|\mathbf{w}|^2 - |\mathbf{v} \times \mathbf{w}|^2 = x_1^2x_2^2 + y_1^2y_2^2 + z_1^2z_2^2 + 2x_1x_2y_1y_2 + 2x_1x_2z_1z_2 + 2y_1y_2z_1z_2 = (\mathbf{v} \cdot \mathbf{w})^2.\]

**Magnitude of the Cross Product**

\[|\mathbf{v}|^2|\mathbf{w}|^2 - |\mathbf{v} \times \mathbf{w}|^2 = (\mathbf{v} \cdot \mathbf{w})^2.\]

So, \(|\mathbf{v} \times \mathbf{w}|^2 = |\mathbf{v}|^2|\mathbf{w}|^2 - (\mathbf{v} \cdot \mathbf{w})^2 = |\mathbf{v}|^2|\mathbf{w}|^2 - (|\mathbf{v}||\mathbf{w}||\cos \theta|^2 = |\mathbf{v}|^2|\mathbf{w}|^2(1 - \cos^2 \theta) = |\mathbf{v}|^2|\mathbf{w}|^2\sin^2 \theta.\]

It follows that \(|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}| |\mathbf{w}| \sin \theta.\)

**Immediate Applications**

- If we place the initial points of vectors \(\mathbf{v}\) and \(\mathbf{w}\) at the same place, we get a parallelogram with the two vectors forming two of the sides and the area will be \(|\mathbf{v} \times \mathbf{w}|.\)
If we place the points of vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ at the same place, we get a parallelopiped with the three vectors forming three of the edges and the volume will be $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. 