Mathematics 116 <u>SOLUTIONS</u>
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1. Write down an effective strategy for determining whether an infinite series converges. *Explain the strategy using plain language, using complete sentences and avoiding the use of mathematical notation. Indeed, note that if you are tempted to use mathematical notation, then you are not writing down a strategy.*

(2-10): Determine whether the indicated improper integral or infinite series converges. In the case of a convergent infinite series, determine whether the convergence is absolute or conditional. Justify your conclusion.

2. $\displaystyle\int_2^\infty \frac{\sqrt{x}}{x^2-1}\,dx$

   Solution: $\dfrac{\sqrt{x}}{x^2-1} < \dfrac{1}{x^{3/2}}$ when x is large and $\displaystyle\int_2^\infty \frac{1}{x^{3/2}}\,dx < \infty$ by the P-Test, so $\displaystyle\int_2^\infty \frac{\sqrt{x}}{x^2-1}\,dx < \infty$ by the Comparison Test.

3. $\sum_{n=2}^\infty \dfrac{(-1)^n\sqrt{n}}{n^2-1}$

   Solution: $\dfrac{\sqrt{n}}{n^2-1} < \dfrac{1}{n^{3/2}}$ when n is large and $\sum_{n=2}^\infty \dfrac{1}{n^{3/2}} < \infty$ by the P-Test, so $\sum_{n=2}^\infty \dfrac{\sqrt{n}}{n^2-1} < \infty$ by the Comparison Test. Hence $\sum_{n=2}^\infty \dfrac{(-1)^n\sqrt{n}}{n^2-1}$ is absolutely convergent.

   Note: We could have used the Integral Test along with the solution to the previous question rather than using the Comparison Test.

4. $\displaystyle\int_0^\infty \frac{x}{x^2+1}\,dx$

   Solution: $\dfrac{x}{x^2+1} > \dfrac{1}{2}\cdot\dfrac{1}{x}$ when x is large and $\displaystyle\int_0^\infty \frac{1}{x}\,dx = \infty$ by the P-Test, so $\displaystyle\int_0^\infty \frac{x}{x^2+1}\,dx = \infty$ by the Comparison Test.

5. $\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+1}$

Solution: $\frac{n}{n^2+1} > \frac{1}{2} \cdot \frac{1}{n}$ when n is large and $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ by the P-Test, so $\sum_{n=0}^{\infty} \frac{n}{n^2+1}\, dn = \infty$ by the Comparison Test. Hence $\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+1}$ is not absolutely convergent.

However, clearly $\frac{n}{n^2+1}$ decreases monotonically to 0, so $\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges by the Alternating Series Test.

One can show $\frac{n}{n^2+1}$ *decreases monotonically to 0 by showing* $\frac{n}{n^2+1} > \frac{n+1}{[n+1]^2+1}$, *which is equivalent to each of the following inequalities, the first of which is obtained by multiplying both sides by the product of the denominators:*

$n[(n+1)^2+1] > (n+1)(n^2+1)$
$n^3+2n^2+2n > n^3+n^2+n+1$
$n^2+n > 1.$

The last is obviously true when $n \geq 1$, *so all are true for positive* n.

Alternatively, one could consider the function $f(x) = \frac{x}{x^2+1}$ *and observe* $f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0$ *when* $x > 1$, *so* $f(x)$ *must be a decreasing function and thus* $\frac{n}{n^2+1}$ *must be a decreasing sequence.*

6. $\int_2^{\infty} \frac{1}{\ln x}\, dx$

Solution: $\frac{1}{\ln x} > \frac{1}{x}$ and $\int_2^{\infty} \frac{1}{x}\, dx = \infty$ by the P-Test, so $\int_2^{\infty} \frac{1}{\ln x}\, dx = \infty$ by the Comparison Test.

7. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

Solution: $\frac{1}{\ln n} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n} = \infty$ by the P-Test, so $sum_{n=2}^{\infty} \frac{1}{\ln n} = \infty$ by the Comparison Test. Hence $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is not absolutely convergent.

However, $\frac{1}{\ln n}$ is clearly monotonically decreasing to 0, so $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is convergent by the Alternating Series Test and is thus conditionally convergent.

8. $\int_0^{\infty} \frac{1+\cos x}{e^x}\, dx$

Solution: $\left| \frac{1+\cos x}{e^x} \right| < 2 \cdot \frac{1}{e^x}$ and we know $\int_0^{\infty} \frac{1}{e^x}\, dx = \int_0^{\infty} e^{-x}\, dx < \infty$, so $\int_0^{\infty} \frac{1+\cos x}{e^x}\, dx < \infty$ by the Comparison Test.

9. $\sum_{n=0}^{\infty} \frac{(-1)^n(1+\cos n)}{2^n}$

Solution: $\left|\frac{(-1)^n(1+\cos n)}{2^n}\right| \leq \frac{2}{2^n}$ and $\sum_{n=0}^{\infty} \frac{2}{2^n}$ is a geometric series with common ration $\frac{1}{2} < 1$, hence convergent, so by the Comparison Test, $\sum_{n=0}^{\infty} \left|\frac{(-1)^n(1+\cos n)}{2^n}\right| < \infty$ and $\sum_{n=0}^{\infty} \frac{(-1)^n(1+\cos n)}{2^n}$ is absolutely convergent.

10. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

Solution: Letting $a_n = \left|\frac{(-1)^n 2^n}{n!}\right| = \frac{2^n}{n!}$, we have $\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} = \frac{2}{1} \cdot \frac{1}{n+1} = \frac{2}{n+1} \to 0 < 1$ as $n \to \infty$, so $\sum_{n=0}^{\infty} \left|\frac{(-1)^n 2^n}{n!}\right| < \infty$ by the Ratio Test. Hence $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$ is absolutely convergent.

11. Find the fifth degree Taylor Polynomial $T_5(x)$ centered at 0 for the function $(x^2-4)^3$.

Solution: One can use the general formula for Taylor Polynomials, but one may also multiply out $(x^2-4)^3 = x^6 - 12x^4 + 48x^2 - 64$ and recall that a Taylor Polynomial has the same terms, up to its degree, as the polynomial it is approximating, so $T_5(x) = -64 + 48x^2 - 12x^4$.

(12-16): Consider the polar curve $r = 1 + \sin(4\theta)$, $0 \leq \theta \leq 2\pi$.

12. Sketch its graph.

    Solution: The graph looks a lot like a four petal rose, with r reaching its maximum of 2 when $\theta = \pi/8, 5\pi/8, 9\pi/8$ and $13\pi/8$.

13. Find the area of the region enclosed by the curve.

    Solution: The area $A = \frac{1}{2}\int_0^{2\pi} (1 + \sin(4\theta))^2\, d\theta = \dfrac{1}{2}\int_0^{2\pi} 1 + 2\sin 4\theta + \sin^2 4\theta\, d\theta$.

    We could find indefinite integrals and apply the Fundamental Theorem of Calculus, but we'll get sneaky, split up the integral, and make each of the following obvious observations:

    $\displaystyle\int_0^{2\pi} 1\, d\theta = 2\pi$

    $\displaystyle\int_0^{2\pi} \sin 4\theta\, d\theta = 0$. *If it's not obvious, look at the graph of* $y = \sin 4\theta$.

    From their graphs, it's obvious $\displaystyle\int_0^{2\pi} \sin^2 4\theta\, d\theta = \int_0^{2\pi} \cos^2 4\theta\, d\theta$, so $\displaystyle\int_0^{2\pi} \sin^2 4\theta\, d\theta = \frac{1}{2}\int_0^{2\pi} \sin^2 4\theta + \cos^2 4\theta\, d\theta = \frac{1}{2}\int_0^{2\pi} 1\, d\theta = \frac{1}{2} \cdot 2\pi = \pi$.

    Hence, $\frac{1}{2}\displaystyle\int_0^{2\pi} 1 + 2\sin 4\theta + \sin^2 4\theta\, d\theta = \frac{1}{2}(2\pi + 2 \cdot 0 + \pi) = 3\pi/2$.

    So the area is $3\pi/2$.

14. Write down an integral whose value is the length of the curve. *Do not evaluate the integral.*

    Solution: The length is equal to $\displaystyle\int_0^{2\pi} \sqrt{(1 + \sin 4\theta)^2 + 16\cos^2 4\theta}\, d\theta$.

15. Write down an integral whose value is the area of the surface obtained by rotating the curve about the x-axis. *Do not evaluate the integral.*

    Solution: The surface area is equal to $2\pi\displaystyle\int_0^{2\pi} (1+\sin 4\theta)\sin\theta\sqrt{(1 + \sin 4\theta)^2 + 16\cos^2 4\theta}\, d\theta$.

16. Write down an integral whose value is the area of the surface obtained by rotating the curve about the y-axis. *Do not evaluate the integral.*

    Solution: The surface area is equal to $2\pi\displaystyle\int_0^{2\pi} (1+\sin 4\theta)\cos\theta\sqrt{(1 + \sin 4\theta)^2 + 16\cos^2 4\theta}\, d\theta$.

Extra Credit: An American quarter has a diameter of approximately 24 millimeters, with the tip of George Washington's head lying approximately 2 millimeters from the edge of the coin. Suppose a quarter is placed on its edge with the tip of our first president's head directly above the center of the quarter and rolled along a straight line at a speed of 3 centimeters per second. Set up an appropriate coordinate system along with appropriate variables and use parametric equations to express the position of the tip of George's head in terms of the amount of time the coin has been rolling. Assume the dimensions given are exact and the coin will roll forever.

Solution: Imagine the coin in the xy-plane, rolling along the x-axis at a speed of 3 centimeters per second, starting from the origin. Scale the axes in millimeters. (We could use centimeters, but millimeters are more convenient.)

The center of the coin will start at the point $(0, 12)$ and will be at the point $(30t, 12)$ at time t, where t is measured in seconds.

Let us denote by θ the angle through which the radius going through the tip of George Washington's head has moved. (This may also be thought of as the angle between the radius from the center through the tip of George's head and the radius going from the center in the direction of the positive y-axis.)

Since the arc swept out by θ is also the distance $30t$ the coin has rolled and the coin has radius 12, it follows that $\theta = \frac{30t}{12} = \frac{5t}{2}$.

If we draw the triangle formed by a vertical radius, the radius through the tip of George Washington's head and the horizontal line segment from the tip of George Washington's head to the vertical radius, we see the tip of George Washington's head lies $10 \sin\theta$ units to the right and $10 \cos\theta$ units above the center of the coin. *As usual, a negative distance to the right really represents a distance to the left and a negative distance above really represents a distance below. If one draws a picture, it is clear this comes out correctly even when the* sin *and/or* cos *is negative!*

Since the horizontal distance the coin has travelled is $30t$, the tip of George Washington's head lies $30t + 10 \sin\theta$ millimeters to the right of the y-axis. Since the center of the coin is 12 units above the x-axis, the tip of George Washington's head lies $12 + 10 \cos\theta$ millimeters above the x-axis.

We thus have: $x = 30t + 10 \sin\theta$, $y = 12 + 10 \cos\theta$, $\theta = \frac{5t}{2}$.

Writing this as a pair of parametric equations in terms of the amount of time the coin has been rolling, we have:
$x = 30t + 10 \sin(5t/2)$
$y = 12 + 10 \cos(5t/2)$
$t \geq 0$.