Mathematics 116 SOLUTIONS

Professor Alan H. Stein

Due Friday, March 28 This problem set will be graded on the basis of 100 points but will be worth 50 points.

1. Write down an effective strategy for calculating derivatives. The strategy should work for any function built from basic, elementary functions as long as one is familiar with the basic differentiation formulas, including the product, quotient and chain rules and the formulas for derivatives of basic, elementary functions. *Explain the strategy using plain language, using complete sentences and avoiding the use of mathematical notation. Indeed, note that if you are tempted to use mathematical notation, then you are not writing down a strategy.*

2. Write down an effective strategy for calculating integrals. *Explain the strategy using plain language, using complete sentences and avoiding the use of mathematical notation. Indeed, note that if you are tempted to use mathematical notation, then you are not writing down a strategy.*

3. Calculate $\lim_{x\to 0}(1-\cos x)\cot^2 x$. *Extra Credit: Find the limit two completely different ways.*

   Solution: We give two solutions, with or without using L'Hôpital's Rule.

   $\lim_{x\to 0}(1-\cos x)\cot^2 x = \lim_{x\to 0}(1-\cos x)\cdot\dfrac{1+\cos x}{1+\cos x}\cdot\dfrac{\cos^2 x}{\sin^2 x} = \lim_{x\to 0}\dfrac{(1-\cos^2 x)(\cos^2 x)}{(1+\cos x)\sin^2 x} = \lim_{x\to 0}\dfrac{(\sin^2 x)(\cos^2 x)}{(1+\cos x)\sin^2 x} = \lim_{x\to 0}\dfrac{\cos^2 x}{1+\cos x} = \dfrac{1}{2}$.

   Alternatively: $\lim_{x\to 0}(1-\cos x)\cot^2 x = \lim_{x\to 0}\dfrac{1-\cos x}{\tan^2 x} = \lim_{x\to 0}\dfrac{\sin x}{2\tan x\sec^2 x} = \lim_{x\to 0}\dfrac{\sin x}{2\tan x} = \lim_{x\to 0}\dfrac{\sin x}{2\sin x/\cos x} = \lim_{x\to 0}(\sin x)\cdot\dfrac{\cos x}{2\sin x} = \lim_{x\to 0}\dfrac{\cos x}{2} = \dfrac{1}{2}$.

4. Perform a partial fractions decomposition for $\frac{3x-2}{x^2-5x+6}$. *Extra Credit: Find its integral.*

   Solution: $\frac{3x-2}{x^2-5x+6} = \frac{3x-2}{(x-2)(x-3)} = \frac{a}{x-2} + \frac{b}{x-3} = \frac{a(x-3)+b(x-2)}{x^2-5x+6}$, so we must have $a(x-3)+b(x-2) = 3x-2$.

   Letting $x = 2$, we get $-a = 4$, $a = -4$.

   Letting $x = 3$, we get $b = 7$.

   Thus, $\frac{3x-2}{x^2-5x+6} = \frac{-4}{x-2} + \frac{7}{x-3} = \frac{7}{x-3} - \frac{4}{x-2}$

   Extra Credit: $\int \frac{3x-2}{x^2-5x+6}\,dx = \int \frac{7}{x-3} - \frac{4}{x-2}\,dx = 7\ln|x-3| - 4\ln|x-2| + k.$

5. Perform a partial fractions decomposition for $\dfrac{x^2-11x+7}{x^3+5x^2+4x+20}$. *Extra Credit: Find its integral.*

Solution: Looking at the divisors of 20, we find -5 is a zero of $x^3+5x^2+4x+20$, so $x+5$ is a factor. We may use long division or any other legitimate method to find $x^3+5x^2+4x+20=(x+5)(x^2+4)$. Clearly, x^2+4 cannot be factored further.

We thus have $\dfrac{x^2-11x+7}{x^3+5x^2+4x+20}=\dfrac{a}{x+5}+\dfrac{bx+c}{x^2+4}=\dfrac{a(x^2+4)+(bx+c)(x+5)}{x^3+5x^2+4x+20}$. It follows that $a(x^2+4)+(bx+c)(x+5)=x^2-11x+7$.

Letting $x=-5$, we get $29a=87$, $a=3$. Unfortunately, this is as far as this particular method can take us, so we'll find b and c by equating coefficients.

The coefficient of x^2 on the left is $a+b$, so we have $a+b=1$. Since $a=3$, we get $3+b=1$, $b=-2$.

The constant term on the left is $4a+5c$, so we have $4a+5c=7$, $4\cdot 3+5c=7$, $12+5c=7$, $5c=-5$, $c=-1$.

Hence, $\dfrac{x^2-11x+7}{x^3+5x^2+4x+20}=\dfrac{3}{x+5}+\dfrac{-2x-1}{x^2+4}=\dfrac{3}{x+5}-\dfrac{2x+1}{x^2+4}$.

Extra Credit: $\int \dfrac{x^2-11x+7}{x^3+5x^2+4x+20}\,dx=\int \dfrac{3}{x+5}-\dfrac{2x+1}{x^2+4}\,dx=$
$3\int \dfrac{1}{x+5}\,dx-\int \dfrac{2x}{x^2+4}\,dx-\int \dfrac{1}{x^2+4}\,dx.$

Using the substitution $u=x+5$, *we get* $\int \dfrac{1}{x+5}\,dx=\ln|x+5|$.

Using the substitution $u=x^2+4$, *we get* $\int \dfrac{2x}{x^2+4}\,dx=\ln(x^2+4)$.

We might be able to guess that $\int \dfrac{1}{x^2+4}\,dx=\dfrac{1}{2}\arctan(x/2)$. *Alternatively, we might use a substitution* $u=x/2$ *(suggested by writing* $x^2+4=4(1+(x/2)^2)$*) or we might use a trigonometric substitution.*

Hence, $\int \dfrac{x^2-11x+7}{x^3+5x^2+4x+20}\,dx=3\ln|x+5|-\ln(x^2+4)-\tfrac{1}{2}\arctan(x/2)+k$.

6. Calculate $\int \sin^2 x\cos^2 x\,dx$.

Solution: Since both $\sin x$ and $\cos x$ occur to even powers, we use a double angle formula.

$\int \sin^2 x\cos^2 x\,dx=\int \dfrac{1-\cos 2x}{2}\cdot\dfrac{1+\cos 2x}{2}\,dx=\tfrac{1}{4}\int(1-\cos^2 2x)\,dx=\tfrac{1}{4}\int \sin^2 2x\,dx=$
$\tfrac{1}{4}\int \dfrac{1-\cos 4x}{2}\,dx=\tfrac{1}{8}\int 1-\cos 4x\,dx=\tfrac{1}{8}\left(x-\tfrac{\sin 4x}{4}\right)+k.$

7. Calculate $\int \dfrac{x^3}{\sqrt{25-x^2}}\,dx$.

   Solution: We show two methods.

   Using a trigonometric substitution, draw a right triangle with acute angle θ, hypotenuse 5, let the opposite leg equal x and the adjacent leg equal $\sqrt{25-x^2}$.

   $\cos\theta = \dfrac{\sqrt{25-x^2}}{5}$, so $\sqrt{25-x^2} = 5\cos\theta$.

   $\sin\theta = \dfrac{x}{5}$, so $x = 5\sin\theta$, $\dfrac{dx}{d\theta} = 5\cos\theta$, $dx = 5\cos\theta\,d\theta$.

   We thus get $\int \dfrac{x^3}{\sqrt{25-x^2}}\,dx = \int \dfrac{(5\sin\theta)^3}{5\cos\theta}\cdot 5\cos\theta\,d\theta = 125\int \sin^3\theta\,d\theta$.

   Since $\sin\theta$ occurs to an odd power, we substitute $u=\cos\theta$, so $\dfrac{du}{d\theta} = -\sin\theta$, $d\theta = -\dfrac{du}{\sin\theta}$, obtaining $\int \dfrac{x^3}{\sqrt{25-x^2}}\,dx = 125\int \sin^3\theta\cdot\left(-\dfrac{du}{\sin\theta}\right) = -125\int \sin^2\theta\,du =$
   $-125\int(1-\cos^2\theta)\,du = -125\int(1-u^2)\,du = -125(u-u^3/3) = 125(u^3/3-u) =$
   $125(\cos^3\theta/3-\cos\theta) =$
   $125((\sqrt{25-x^2}/5)^3/3 - \sqrt{25-x^2}/5) = \sqrt{25-x^2}^3/3 - 25\sqrt{25-x^2} =$
   $(25-x^2)\sqrt{25-x^2}/3 - 25\sqrt{25-x^2} = -(50+x^2)\sqrt{25-x^2}/3 + k$.

   Alternatively, we might just substitute $u = 25-x^2$, $\dfrac{du}{dx} = -2x$, $dx = -\dfrac{du}{2x}$, so $\int \dfrac{x^3}{\sqrt{25-x^2}}\,dx = \int \dfrac{x^3}{\sqrt{u}}\left(-\dfrac{du}{2x}\right) = -\tfrac{1}{2}\int \dfrac{x^2}{\sqrt{u}}\,du$.

   Since $u = 25-x^2$, $x^2 = 25-u$, so $\int \dfrac{x^3}{\sqrt{25-x^2}}\,dx = -\tfrac{1}{2}\int \dfrac{25-u}{\sqrt{u}}\,du =$
   $-\tfrac{1}{2}\int 25u^{-1/2} - u^{1/2}\,du = -\tfrac{1}{2}\left(25u^{1/2}/(1/2) - u^{3/2}/(3/2)\right) = \sqrt{u}(u/3-25) =$
   $\sqrt{25-x^2}\left(\dfrac{25-x^2}{3} - 25\right) = -(50+x^2)\sqrt{25-x^2}/3 + k$.

8. Calculate $\int \dfrac{x+2}{x+1}\,dx$.

   Solution: Substituting $u = x+1$, $\dfrac{du}{dx} = 1$, $dx = du$, we get $\int \dfrac{x+2}{x+1}\,dx = \int \dfrac{x+2}{u}\,du =$ $\int \dfrac{u+1}{u}\,du = \int 1 + \tfrac{1}{u}\,du = u + \ln|u| = x+1+\ln|x+1| = x + \ln|x+1| + k$.

9. Calculate $\int \sin^2 x\cos^3 x\,dx$.

   Solution: Since $\cos x$ occurs to an odd power, we substitute $u = \sin x$, $\dfrac{du}{dx} = \cos x$, $dx = \dfrac{du}{\cos x}$, so $\int \sin^2 x\cos^3 x\,dx = \int u^2\cos^3 x\cdot\dfrac{du}{\cos x} = \int u^2\cos^2 x\,du = \int u^2(1-\sin^2 x)\,du =$ $\int u^2(1-u^2)\,du = \int u^2 - u^4\,du = u^3/3 - u^5/5 = \sin^3 x/3 - \sin^5 x/5 + k$.

10. Calculate $\int \arctan(5x)\,dx$.

Solution: Integrating by parts, let $f(x) = \arctan(5x)$, $g'(x) = 1$, so $f'(x) = \dfrac{5}{1+25x^2}$, $g(x) = x$ and $\int \arctan(5x)\,dx = \arctan(5x)\cdot x - \int \dfrac{5}{1+25x^2}\cdot x\,dx =$ $x\arctan(5x) - 5\int \dfrac{x}{1+25x^2}\,dx$.

We may substitute $u = 1+25x^2$, $\dfrac{du}{dx} = 50x$, $dx = \dfrac{du}{50x}$ in the latter integral to get $\int \arctan(5x)\,dx = x\arctan(5x) - 5\int \dfrac{x}{u}\cdot\dfrac{du}{50x} = x\arctan(5x) - \frac{1}{10}\int \frac{1}{u}\,du =$ $x\arctan(5x) - \frac{1}{10}\ln|u| = x\arctan(5x) - \frac{1}{10}\ln(1+25x^2) + k$.

11. Calculate $\int \dfrac{\ln x}{x^2}\,dx$.

Solution: Using integration by parts, let $f(x) = \ln x$, $g'(x) = x^{-2}$, so $f'(x) = \frac{1}{x}$, $g(x) = -\frac{1}{x}$ and $\int \dfrac{\ln x}{x^2}\,dx = (\ln x)(-\frac{1}{x}) - \int \frac{1}{x}\cdot(-\frac{1}{x})\,dx = -\frac{\ln x}{x} + \int x^{-2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + k$

12. Calculate $\int t^2 \sin t\,dt$.

Solution: Using integration by parts, let $f(t) = t^2$, $g'(t) = \sin t$, so $f'(t) = 2t$, $g(t) = -\cos t$ and $\int t^2\sin t\,dt = t^2(-\cos t) - \int 2t(-\cos t)\,dt = -t^2\cos t + 2\int t\cos t\,dt$.

Use integration by parts again on the latter integral, this time letting $f(t) = t$, $g'(t) = \cos t$, so $f'(t) = 1$, $g(t) = \sin t$ and $\int t^2\sin t\,dt = -t^2\cos t + 2(t\sin t - \int 1\cdot\sin t\,dt) =$ $-t^2\cos t + 2t\sin t - 2\int \sin t\,dt = -t^2\cos t + 2t\sin t + 2\cos t + k$.

13. Calculate $\int \frac{x}{\sqrt{25-x^2}}\,dx$.

Solution: We show three methods, starting with the hardest and ending with the easiest.

Using a trigonometric substitution, draw a right triangle with acute angle θ, hypotenuse 5, let the opposite leg equal x and the adjacent leg equal $\sqrt{25-x^2}$.

$\cos\theta = \frac{\sqrt{25-x^2}}{5}$, so $\sqrt{25-x^2} = 5\cos\theta$.

$\sin\theta = \frac{x}{5}$, so $x = 5\sin\theta$, $\frac{dx}{d\theta} = 5\cos\theta$, $dx = 5\cos\theta\,d\theta$.

We thus get $\int \frac{x}{\sqrt{25-x^2}}\,dx = \int \frac{5\sin\theta}{5\cos\theta}\cdot 5\cos\theta\,d\theta = 5\int\sin\theta\,d\theta = -5\cos\theta =$
$-5\cdot\frac{\sqrt{25-x^2}}{5} = -\sqrt{25-x^2}+k$.

Alternatively, we might just substitute $u = 25-x^2$, $\frac{du}{dx} = -2x$, $dx = -\frac{du}{2x}$, so $\int \frac{x}{\sqrt{25-x^2}}\,dx = \int \frac{x}{\sqrt{u}}\cdot\left(-\frac{du}{2x}\right) = -\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}u^{1/2}/(1/2) = -\sqrt{25-x^2}+k$.

Finally, we might guess the integral is something like $\sqrt{25-x^2}$. Differentiating, we get $\frac{d}{dx}\left(\sqrt{25-x^2}\right) = -\frac{x}{\sqrt{25-x^2}}$. Since this is off by a minus sign, we conclude $\int \frac{x}{\sqrt{25-x^2}}\,dx = -\sqrt{25-x^2}+k$.