Mathematics 116 — SOLUTIONS

Professor Alan H. Stein

Due Friday, March 8 This problem set will be graded on the basis of 100 points but will be worth 50 points.

1. Calculate each of the following derivatives.

   (a) $\frac{d}{dx}(\sin(\ln x))$.

   Solution: $\frac{d}{dx}(\sin(\ln x)) = \cos(\ln x) \cdot \frac{1}{x} =$ cos ln

(j) $\frac{d}{dx}(\ln(\ln x))$.

Solution: $\frac{d}{dx}(\ln(\ln x)) = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}$.

(k) $\frac{d}{dx}(\exp(\ln x))$.

Solution: $\frac{d}{dx}(\exp(\ln x)) = \frac{d}{dx}(x) = 1$.

(l) $\frac{d}{}$

2. A population of rabbits grows exactly 5 percent per year. How long does it take for the population to triple? *Ignore the reality that the population is a discrete function of time and use an exponential model to obtain a reasonable answer.*

   Solution: Let x be the population of rabbits and t represent time, measured in years. We know $x = ae^{bt}$ for some constants $a, b \in \mathbb{R}$.

   If the initial population is x_0, we know $x = x_0$ when $t = 0$, so $x_0 = ae^{b \cdot 0} = a$, so $x = x_0 e^{bt}$.

   If the population grows 5 percent each year, we know $x = 1.05x_0$ when $t = 1$, so $1.05x_0 = x_0 e^{b \cdot 1}$. Solving for b, we get: $e^b = 1.05$, $b = \ln 1.05$, so $x = x_0 e^{t \ln 1.05}$.

   We need to find when $x = 3x_0$, so we solve $3x_0 = x_0 e^{t \ln 1.05}$ for t. Solving, we get: $e^{t \ln 1.05} = 3$, $t \ln 1.05 = \ln 3$, $t = \dfrac{\ln 3}{\ln 1.05}$.

   It thus takes $\dfrac{\ln 3}{\ln 1.05}$ years for the population to triple.
   This is approximately 22.5170853055 years.

3. A fossil contains 3 percent of the amount of a radioactive substance it had when it was part of a living organism. Assuming the radioactive substance has a half-life of 247 years, find the age of the fossil. *Your solution should be self-contained, assuming only that the amount of a radioactive substance is an exponential function of time.*

   Solution: Let x represent the mass of the radioactive substance and let t represent time, measured in years, starting from the moment the organism transitioned from life to death. We know $x = ae^{bt}$ for some constants $a, b \in \mathbb{R}$.

   If we let x_0 represent the amount of the substance present when the organism died, then $x = x_0$ when $t = 0$, so $x_0 = ae^{b \cdot 0}$. Simplifying, we get $x_0 = a$, so $x = x_0 e^{bt}$.

   Since the half-life is 247 years, we know $x = \dfrac{1}{2}x_0$ when $t = 247$, so $\frac{1}{2}x_0 = x_0 e^{b \cdot 247}$. Solving for b, we get: $e^{247b} = \frac{1}{2}$, $247b = \ln(1/2) = -\ln 2$, $b = -\frac{\ln 2}{247}$. Thus $x = x_0 e^{-(t \ln 2)/247}$.

   Since the fossil now contains 3 percent of the substance it did when the organism passed away, we have $0.03x_0 = x_0 e^{-(t \ln 2)/247}$. Solving for t: $e^{-(t \ln 2)/247} = 0.03$, $-\frac{t \ln 2}{247} = \ln 0.03$, $t = \frac{247 \ln 0.03}{\ln 2}$.

   The fossil is thus $-\frac{247 \ln 0.03}{\ln 2}$ years old, or approximately $1,249.5467412$ years old.

4. Define the arctan function and use that definition to show $\dfrac{d}{dx}(\arctan x)$ must be $\dfrac{1}{1+x^2}$.

   Solution: $\arctan = \mathrm{Tan}^{-1}$, where Tan is the principal tangent function, defined as the restriction of the ordinary tan function to the interval $(-\pi/2, \pi/2)$.

   Thus, if $y = \arctan x$, then $x = tany$. Di�erentiating implicitly, we have $\dfrac{d}{dx}(x) = \dfrac{d}{dx}(\tan y)$, $1 = \sec^2 y \dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \frac{1}{\sec^2 y}$.

   Since $\sec^2 y = 1 + \tan^2 y = 1 + x^2$, we get $\dfrac{dy}{dx} = \frac{1}{1+x^2}$, so $\dfrac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$.

(5-10: Calculate the following limits. Extra credit will be given for any computation correctly done without the use of L'Hôpital's Rule.

5. $\lim_{x\to 0} \dfrac{\sin 6x}{\sin 2x}$

   Solution: Using L'Hôpital's Rule, $\lim_{x\to 0} \dfrac{\sin 6x}{\sin 2x} = \lim_{x\to 0} \dfrac{(\cos 6x)\cdot 6}{(\cos 2x)\cdot 2}$. Since both $\cos 6x$ and $\cos 2x$ obviously approach 1 as $x \to 0$, we get $\lim_{x\to 0} \dfrac{\sin 6x}{\sin 2x} = \frac{6}{2} = 3$.

   Without L'Hôpital's Rule, $\lim_{x\to 0} \dfrac{\sin 6x}{\sin 2x} = \lim_{x\to 0} 3\dfrac{(\sin 6x)/(6x)}{(\sin 2x)(2x)} = 3$.

6. $\lim_{x\to\infty} \dfrac{(\ln x)^2}{x}$

   Solution: Using L'Hôpital's Rule, $\lim_{x\to\infty} \dfrac{(\ln x)^2}{x} = \lim_{x\to\infty} \dfrac{(2\ln x)\cdot \frac{1}{x}}{1} = \lim_{x\to\infty} 2\dfrac{\ln x}{x} = 0$, since we've shown $\lim_{x\to\infty} \dfrac{\ln x}{x} = 0$.

7. $\lim_{x\to 0^+} (\sin x)\ln x$

   Solution: $\lim_{x\to 0^+} (\sin x)\ln x = \lim_{x\to 0^+} \dfrac{\ln x}{\csc x} = \lim_{x\to 0^+} \dfrac{1/x}{-\csc x \cot x} =$
   $-\lim_{x\to 0^+} \dfrac{1}{x\cdot \frac{1}{\sin x}\cdot \frac{\cos x}{\sin x}} = -\lim_{x\to 0^+} \dfrac{\sin^x}{x\cos x} = -\lim_{x\to 0^+} \dfrac{\sin x}{x}\dfrac{\sin x}{\cos x} =$
   $--\lim_{x\to 0^+} \dfrac{\sin x}{x} \lim_{x\to 0^+} \dfrac{\sin x}{\cos x} = -1\cdot 0 = 0.$

   Alternatively, without the use of L'Hôpital's Rule:

   $\lim_{x\to 0^+} (\sin x)\ln x = \lim_{x\to 0^+} \dfrac{\sin x}{x}\cdot x\ln x = \lim_{x\to 0^+} \dfrac{\sin x}{x}\cdot \lim_{x\to 0^+} x\ln x = 1\cdot 0 = 0$, since we have earlier found these two limits.

8. $\lim_{x\to\infty}\left(\frac{2x+3}{2x}\right)^x$

   Solution: $\lim_{x\to\infty}\left(\frac{2x+3}{2x}\right)^x = \lim_{x\to\infty}\exp(x\ln(1+3/(2x)))$
   $= \exp(\lim_{x\to\infty} x\ln(1+3/(2x)))$.

   Since $\lim_{x\to\infty} x\ln(1+3/(2x)) = \lim_{x\to\infty}\frac{ln(1+3/(2x))}{1/x} = \lim_{x\to\infty}\frac{\frac{1}{1+3/(2x)}\cdot\frac{3}{2}\cdot(-1/x^2)}{-1/x^2} =$
   $\lim_{x\to\infty}\frac{1}{1+3/(2x)}\cdot\frac{3}{2} = \frac{3}{2}$, it follows that $\lim_{x\to\infty}\left(\frac{2x+3}{2x}\right)^x = \exp(3/2) = e^{3/2}$.

   Alternatively, $\lim_{x\to\infty}\left(\frac{2x+3}{2x}\right)^x = \lim_{x\to\infty}\left[\left(1+\frac{1}{2x/3}\right)^{2x/3}\right]^{3/2} = e^{3/2}$.

9. $\lim_{x\to 0}\sin(8x)\cot(2x)$

   Solution: Using L'Hôpital's Rule: $\lim_{x\to 0}\sin(8x)\cot(2x) = \lim_{x\to 0}\frac{\sin 8x}{\tan 2x} =$
   $\lim_{x\to 0}\frac{(\cos 8x)\cdot 8}{(\sec^2 2x)\cdot 2} = 4$, since both $\cos 8x$ and $\sec 2x$ approach 1 as $x\to 0$.

   Alternatively, $\lim_{x\to 0}\sin(8x)\cot(2x) = \lim_{x\to 0}\frac{\sin 8x}{\sin 2x/\cos 2x} = \lim_{x\to 0}\frac{\sin 8x}{\sin 2x} =$
   $\lim_{x\to 0} 4\cdot\frac{\frac{\sin 8x}{8x}}{\frac{\sin 2x}{2x}} = 4\cdot\frac{1}{1} = 4$.

10. $\lim_{x\to\infty}(\sqrt{x^2+4x}-x)$

    Solution: $\lim_{x\to\infty}(\sqrt{x^2+4x}-x) = \lim_{x\to\infty}(\sqrt{x^2+4x}-x)\cdot\frac{\sqrt{x^2+4x}+x}{\sqrt{x^2+4x}+x} =$
    $\lim_{x\to\infty}\frac{4x}{\sqrt{x^2+4x}+x} = \lim_{x\to\infty}\frac{4x}{x(\sqrt{1+4/x}+1)} = \lim_{x\to\infty}\frac{4}{\sqrt{1+4/x}+1} = \frac{4}{2} = 2$.