Mathematics 116 <u>SOLUTIONS</u>

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Due Friday, February 15 This problem set will be graded on the basis of 100 points but will be worth 50 points.

1. Calculate $\int 6x^2 + 9\sin x - 2\sec^2 x\,dx$.

   Solution: $\int 6x^2 + 9\sin x - 2\sec^2 x\,dx = 2x^3 - 9\cos x - 2\tan x + k$

2. Calculate $\int_{\pi/4}^{\pi/3} \sin x\,dx$. *Remember: Your calculation should be exact, not a calculator approximation.*

   Solution: $\int_{\pi/4}^{\pi/3} \sin x\,dx = -\cos x\Big|_{\pi/4}^{\pi/3} = -\cos(\pi/3) - [-\cos(\pi/4)] = -\frac{1}{2} + \frac{1}{\sqrt{2}}$.

3. Calculate $\int x^3\sqrt{x^4+6}\,dx$.

   Solution: One may guess at $(x^4+6)^{3/2}$. Differentiating, $\dfrac{d}{dx}\left((x^4+6)^{3/2}\right) = \frac{3}{2}(x^4+6)^{1/2}\cdot 4x^3 = 6\sqrt{x^4+6}$, exactly 6 times what is needed. We conclude $\int x^3\sqrt{x^4+6}\,dx = \frac{(x^4+6)^{3/2}}{6}$.

   Alternatively, one may substitute $u = x^4+6$. Since $\dfrac{du}{dx} = 4x^3$, $du = 4x^3\,dx$, $dx = \frac{du}{4x^3}$, so $\int x^3\sqrt{x^4+6}\,dx = \int x^3\sqrt{u}\cdot\frac{du}{4x^3} = \frac{1}{4}\int u^{\frac{1}{2}}\,du = \frac{1}{4}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}} = \frac{1}{6}u^{\frac{3}{2}} = \frac{(x^4+6)^{3/2}}{6}$.

4. Consider the function $f(x) = x^3 + 2x - 3$. Analyze monotonicity and concavity. Find all extrema and points of inflection. Sketch its graph.

   Solution: $f'(x) = 3x^2 + 2 > 0$ everywhere, so f is increasing everywhere.

   $f''(x) = 6x\begin{cases} > 0 & \text{for } x > 0 \\ < 0 & \text{for } x < 0 \end{cases}$. Thus the graph of f is concave up when $x > 0$ and concave down when $x < 0$.

   There are thus no extrema, but the point $(0,-3)$ is a point of inflection.

5. Find the points of intersection of the graphs of $y = x^3 + 2x - 3$ and $y = 23(x-1)$.

   Solution: We find the points of intersection by solving the equations simultaneously.

   Using substitution, we have $x^3 + 2x - 3 = 23x - 23$, $x^3 - 21x + 20 = 0$.

   Looking at the divisors of 20, we easily find $x = 1$ is a solution, so we can factor $x - 1$ from $x^3 - 21x + 20$ to get $x^3 - 21x + 20 = (x-1)(x^2+x-20)$.

   We can easily factor $x^2 + x - 20 = (x+5)(x-4)$ by trial and error, so we have $x^3 - 21x + 20 = (x-1)(x+5)(x-4)$.

   We may thus rewrite the equation in the form $(x-1)(x+5)(x-4) = 0$, and the solutions are clearly $x = 1$, $x = -5$ and $x = 4$.

   The points of intersection are thus $(-5,-138)$, $(1,0)$ and $(4,69)$.

(6-21): Let $\mathcal{D}$ be the region in the first quadrant bounded by the graphs of $y = x^3 + 2x - 3$ and $y = 23(x - 1)$. Let $\mathcal{C}$ be the portion of the graph of $y = x^3 + 2x - 3$ which forms part of the boundary of $\mathcal{D}$. For each question, sketch the relevant curve, plane region, surface or solid and set up a definite integral (or sum or differece of definite integrals) whose value is the quantity requested. *Do not evaluate the definite integrals.* Hint: It is expected you will wind up spending more time sketching the geometrical objects than writing down the definite integrals. These are almost short answer questions.

6. The area of $\mathcal{D}$.

   Solution:

   $\int_1^4 23(x-1) - (x^3 + 2x - 3)\, dx$

7. The volume of the solid obtained by rotating $\mathcal{D}$ about the x-axis.

   Solution: $\pi \int_1^4 [23(x-1)]^2 - [(x^3 + 2x - 3)]^2\, dx$

8. The volume of the solid obtained by rotating $\mathcal{D}$ about the line $y = -3$.

   Solution: $\pi \int_1^4 [23(x-1) + 3]^2 - [(x^3 + 2x - 3) + 3]^2\, dx$

9. The volume of the solid obtained by rotating $\mathcal{D}$ about the line $y = 70$.

   Solution: $\pi \int_1^4 [70 - (x^3 + 2x - 3)]^2 - [70 - 23(x-1)]^2\, dx$

10. The volume of the solid obtained by rotating $\mathcal{D}$ about the y-axis.

    Solution: $2\pi \int_1^4 x[23(x-1) - (x^3 + 2x - 3)]\, dx$

11. The volume of the solid obtained by rotating $\mathcal{D}$ about the line $x = 10$.

    Solution: $2\pi \int_1^4 (10 - x)[23(x-1) - (x^3 + 2x - 3)]\, dx$

12. The volume of the solid obtained by rotating $\mathcal{D}$ about the line $x = 1$.

    Solution: $2\pi \int_1^4 (x-1)[23(x-1) - (x^3 + 2x - 3)]\, dx$

13. The volume of the solid obtained by rotating $\mathcal{D}$ about the line $x = -5$.

    Solution: $2\pi \int_1^4 (x+5)[23(x-1) - (x^3 + 2x - 3)]\, dx$

14. The length of $\mathcal{C}$.

    Solution: Since $\dfrac{d}{dx}(x^3 + 2x - 3) = 3x^2 + 2$, the length is $\int_1^4 \sqrt{1 + (3x^2 + 2)^2}\, dx$.

15. The area of the surface obtained by rotating $\mathcal{C}$ about the x-axis.

    Solution: $2\pi \int_1^4 (x^3 + 2x - 3)\sqrt{1 + (3x^2 + 2)^2}\, dx$.

16. The area of the surface obtained by rotating $\mathcal{C}$ about the line $y = -3$.

    Solution: $2\pi \int_1^4 (x^3 + 2x - 3 + 3)\sqrt{1 + (3x^2 + 2)^2}\, dx = 2\pi \int_1^4 (x^3 + 2x)\sqrt{1 + (3x^2 + 2)^2}\, dx$.

17. The area of the surface obtained by rotating $\mathcal{C}$ about the line $y = 70$.

    Solution: $2\pi \int_1^4 [70 - (x^3 + 2x - 3)]\sqrt{1 + (3x^2 + 2)^2}\, dx$.

18. The area of the surface obtained by rotating $\mathcal{C}$ about the y-axis.

    Solution: $2\pi \int_1^4 x\sqrt{1 + (3x^2 + 2)^2}\, dx$.

19. The area of the surface obtained by rotating $\mathcal{C}$ about the line $x = 10$.

    Solution: $2\pi \int_1^4 (10 - x)\sqrt{1 + (3x^2 + 2)^2}\, dx$.

20. The area of the surface obtained by rotating $\mathcal{C}$ about the line $x = 1$.

    Solution: $2\pi \int_1^4 (x - 1)\sqrt{1 + (3x^2 + 2)^2}\, dx$.

21. The area of the surface obtained by rotating $\mathcal{C}$ about the line $x = -5$.

    Solution: $2\pi \int_1^4 (x + 5)\sqrt{1 + (3x^2 + 2)^2}\, dx$.