Riemann Sums

Partition $P = \{x_0, x_1, \ldots, x_n\}$ of an interval $[a, b]$.

$c_k \in [x_{k-1}, x_k]$

$\mathcal{R}(f, P, a, b) = \sum_{k=1}^{n} f(c_k)\Delta x_k$

As the widths Δx_k of the subintervals approach 0, the Riemann Sums *hopefully* approach a limit, the integral of f from a to b, written $\int_a^b f(x)\,dx$.

Fundamental Theorem of Calculus

Theorem 1 (FTC-Part I). *If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is defined on $[a, b]$ and $F'(x) = f(x)$.*

Theorem 2 (FTC-Part II). *If f is continuous on $[a, b]$ and $F(x) = \int f(x)\,dx$ on $[a, b]$, then $\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$.*

Indefinite Integrals

Indefinite Integral: $\int f(x)\,dx = F(x)$ if and only if $F'(x) = f(x)$.

In other words, the terms indefinite integral and antiderivative are synonymous.

Every di erentiation formula yields an integration formula.

Substitution Rule

For Indefinite Integrals:

If $u = g(x)$, then $\int f(g(x))g'(x)\,dx = \int f(u)\,du$.

For Definite Integrals:

If $u = g(x)$, then $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$.

Steps in Mechanically Applying the Substitution Rule

Note: The variables do not have to be called x and u.

(1) Choose a substitution $u = g(x)$.

(2) Calculate $\dfrac{du}{dx} = g'(x)$.

(3) Treat $\dfrac{du}{dx}$ as if it were a fraction, a quotient of di erentials, and solve for dx, obtaining $dx = \dfrac{du}{g'(x)}$.

(4) Go back to the original integral and replace $g(x)$ by u and replace dx by $\dfrac{du}{g'(x)}$. *If it's a definite integral, change the limits of integration to $g(a)$ and $g(b)$.*

Steps in Mechanically Applying the Substitution Rule

(5) Simplify the integral.

(6) If the integral no longer contains the original independent variable, usually x, try to calculate the integral. If the integral still contains the original independent variable, see whether that variable can be eliminated, possibly by solving the equation $u = g(x)$ for x in terms of u, or else try another substitution.

Applications of Definite Integrals

- Areas between curves
- Volumes - starting with solids of revolution
- Arc length
- Surface area
- Work
- Probability

Standard Technique for Applications

(1) Try to estimate some quantity Q.
(2) Note that one can reasonably estimate Q by a Riemann Sum of the form $\sum f(c_k^*)\Delta x_k$ for some function f over some interval $a \le x \le b$.
(3) Conclude that the quantity Q is exactly equal to the definite integral $\int_a^b f(x)\,dx$.

Alternate Technique

Recognize that some quantity can be thought of as a function $Q(x)$ of some independent variable x, try to find the derivative $Q'(x)$, find that $Q'(x) = f(x)$ for some function $f(x)$, and conclude that $Q(x) = \int f(x)\,dx + k$ for some constant k. *Note that this is one way the Fundamental Theorem of Calculus was proven.*

Areas

- The area of the region $\{(x,y)|0 \le y \le f(x), a \le x \le b\}$ is equal to $\int_a^b f(x)\,dx$.
- The area of the region $\{(x,y)|0 \le x \le f(y), a \le y \le b\}$ is equal to $\int_a^b f(y)\,dy$.
- The area of the region $\{(x,y)|f(x) \le y \le g(x), a \le x \le b\}$ is equal to $\int_a^b g(x) - f(x)\,dx$.
- The area of the region $\{(x,y)|f(y) \le x \le g(y), a \le y \le b\}$ is equal to $\int_a^b g(y) - f(y)\,dy$.

Generally: If the cross section perpendicular to the t axis has height $\mathrm{ht}(t)$ for $a \le t \le b$ then the area of the region is $\int_a^b \mathrm{ht}(t)\,dt$.

Volumes

Consider a solid with cross-sectional area $A(x)$ for $a \le x \le b$.

Assume $A(x)$ is a continuous function.

Slice the solid like a salami.

Each slice, of width Δx_k, will have a volume $A(x_k^*)\Delta x_k$ for some $x_{k-1} \le x_k^* \le x_k$.

The total volume will be $\sum_k A(x_k^*)\Delta x_k$, i.e. $\mathcal{R}(f, P, a, b)$.

Conclusion: The volume is $\int_a^b A(x)\,dx$.

Example: Tetrahedron

Example: Solid of Revolution – the cross section is a circle, so the cross sectional area is πr^2, where r is the radius of the circle.

Variation: Cylindrical Shells

Take a plane region $\{(x, y)|0 \le y \le f(x),\ 0 \le a \le x \le b\}$ and rotate the region about the $y-$axis.

Break the original plane region into vertical strips and note that, when rotated around, each vertical strip generates a *cylindrical shell*.

Estimate the volume ΔV_k of a typical cylindrical shell.

$\Delta V_k \approx 2\pi x_k^* f(x_k^*)\Delta x_k$, so the entire volume can be approximated by $\sum_k 2\pi x_k^* f(x_k^*)\Delta x_k = \mathcal{R}(2\pi x f(x), P, a, b)$, so we can conclude that the volume is

$V = 2\pi \int_a^b x f(x)\,dx$.

Arc Length

Problem

Estimate the length of a curve $y = f(x)$, $a \le x \le b$, where f is continuous and differentiable on $[a, b]$.

Solution

(1) Partition the interval $[a, b]$ in the usual way, $a = x_0 \le x_1 \le x_2 \le x_3 \le \cdots \le x_{n-1} \le x_n = b$.

(2) Estimate the length Δs_k of each subinterval, for $x_{k-1} \le x \le x_k$, by the lenght of the line segment connecting $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. Using the Pythagorean Theorem, we get
$\Delta s_k \approx \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1})^2)}$

(3) Using the *Mean Value Theorem*, $f(x_k) - f(x_{k-1}) = f'(x_k^*)(x_k - x_{k-1})$ for some $x_k \in (x_{k-1}, x_k)$.

(4) $\Delta s_k \approx \sqrt{[1 + (f'(x_k^*))^2](x_k - x_{k-1})^2}$
$= \sqrt{1 + (f'(x_k^*))^2}\Delta x_k$, where $\Delta x_k = x_k - x_{k-1}$.

(5) The total length is $\approx \sum_k \sqrt{1 + (f'(x_k^*))^2}\Delta x_k$
$= \mathcal{R}(\sqrt{1 + f'^2}, P, a, b)$.

(6) We conclude that the total length is
$\int_a^b \sqrt{1 + (f'(x))^2}\,dx$.

Heuristics and Alternate Notations

$$
\begin{aligned}
(\Delta s)^2 &\approx (\Delta x)^2 + (\Delta y)^2 \\
(ds)^2 &= (dx)^2 + (dy)^2 \\
\left(\frac{ds}{dx}\right)^2 &= 1 + \left(\frac{dy}{dx}\right)^2 \\
\frac{ds}{dx} &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \\
s &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \\
s &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx
\end{aligned}
$$

Area of a Surface of Revolution

Problem

Estimate the area of a surface obtained by rotating a curve $y = f(x)$, $a \le x \le b$ about the $x-$axis.

Solution

(1) Partition the interval $[a, b]$ in the usual way, $a = x_0 \le x_1 \le x_2 \le x_3 \le \cdots \le x_{n-1} \le x_n = b$.
(2) Estimate the area ΔS_k of the portion of the surface between x_{k-1} and x_k by the product of the length Δs_k of that portion of the curve with the circumference C_k of a circle obtained by rotating a point on that portion of the curve about the $x-$axis.
(3) Estimate Δs_k by $\sqrt{1 + (f'(x_k^*))^2}\Delta x_k$.
(4) Estimate C_k by $2\pi f(x_k^*)$.
(5) $\Delta S_k \approx \sqrt{1 + (f'(x_k^*))^2}\Delta x_k \cdot 2\pi f(x_k^*)$
$= 2\pi f(x_k^*)\sqrt{1 + (f'(x_k^*))^2}\Delta x_k$.
(6) The total surface area may be approximated as $S \approx \sum_k 2\pi f(x_k^*)\sqrt{1 + (f'(x_k^*))^2}\Delta x_k$
$= \mathcal{R}(2\pi f(x)\sqrt{1 + (f'(x))^2}, P, a, b)$.
(7) $S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$.

Variation – Rotating About the $y-$axis

The analysis is essentially the same. The only difference is that the radius of the circle is x_k^* rather than $f(x_k^*)$, so in the formula for the area $f(x)$ simply gets replaced by x and we get
$S = 2\pi \int_a^b x\sqrt{1 + (f'(x))^2}\, dx$.

Work

Problem

An object is moved along a straight line (the $x-$axis) from $x = a$ to $x = b$. The force exerted on the object in the direction of the motion is given by the force function $F(x)$. Find the amount of work done in moving the object.

Work – Solution

If $F(x)$ was just a constant function, taking on a constant value k, one could simply multiply force times distance, getting $k(b-a)$. Since $F(x)$ is not constant, things are more complicated.

It's reasonable to assume that $F(x)$ is a continuous function and does not vary much along a short subinterval of $[a,b]$. So, partition $[a,b]$ in the usual way, $a = x_0 \le x_1 \le x_2 \le x_3 \le \cdots \le x_{n-1} \le x_n = b$, where each subinterval $[x_{k-1}, x_k$ is short enough so that $F(x)$ doesn't change much along it.

Work – Solution

We can thus estimate the work ΔW_k performed along that subinterval by $F(x_k^*)\Delta x_k$, where x_k^* is some point in the interval and $\Delta x_k = (x_k - x_{k-1})$ is the length of the interval. *Indeed, there will be some* x_k^* *for which this will be exactly equal to* ΔW_k.

The total amount of work $W = \sum_{k=1}^{n} \Delta W_k = \sum_{k=1}^{n} F(x_k^*)\Delta x_k = \mathcal{R}(F,P,a,b)$, so we can conclude

$$W = \int_a^b F(x)\,dx.$$

The Natural Logarithm Function

Problem: The formula $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$ has one problem – it doesn't hold for $n = -1$. On the other hand, we know from the Fundamental Theorem of Calculus that $\int \dfrac{1}{x}\,dx$ exists everywhere except at 0.

Solution: Define a function to be that anti-derivative and examine its properties.

The Natural Logarithm Function

Definition 1. $f(x) = \int_1^x \dfrac{1}{t}\,dt$ *for* $x > 0$

By the Fundamental Theorem of Calculus, f is well defined and differentiable for $x > 0$, with $f'(x) = 1/x$. It follows that $f'(x) > 0$

and f is increasing everywhere in the domain of f. It is also fairly immediately clear that $f(x)\begin{cases} < 0 \text{ when } 0 < x < 1 \\ = 0 \text{ when } x = 1 \\ > 0 \text{ when } x > 1. \end{cases}$

We need only find f'' to analyze the concavity of the graph and get a pretty good sketch of it. Since $f'(x) = 1/x$, it follows that $f''(x) = -1/x^2 < 0$ for $x > 0$, so the graph of f is concave down in its entire domain.

Summary

- f defined in the right half plane.
- f is increasing.
- f is concave down.
- $f(x)$ is negative for $0 < x < 1$.
- $f(1) = 0$.
- $f(x)$ is positive for $x > 1$.

Geometrically, it seems obvious that $\lim_{x\to 0^+} f(x) = -\infty$, but it is not clear whether the graph has a horizontal asymptote or $\lim_{x\to\infty} f(x) = \infty$.

Right Hand Limit at 0

Lemma. $\lim_{x\to 0^+} f(x) = -\infty$

The proof will use the following

Claim. *For* $n \in \mathbb{Z}^+$, $f(1/2^n) - f(1/2^{n-1}) < -1/2$.

Proof. $f(1/2^n) - f(1/2^{n-1}) = \int_{1/2^{n-1}}^{1/2^n} \frac{1}{t}\,dt = -\int_{1/2^n}^{1/2^{n-1}} \frac{1}{t}\,dt$.

Since $\frac{1}{t} > 2^{n-1}$ in the interval of integration, it follows that $\int_{1/2^n}^{1/2^{n-1}} \frac{1}{t}\,dt > 2^{n-1}\cdot\frac{1}{2^n} = \frac{1}{2}$ and the conclusion follows immediately. □

Proof of the Lemma

Proof. Let $n \in \mathbb{Z}^+$. Then $f(1/2^n) = f(1/2^n) - f(1) = [f(1/2^n) - f(1/2^{n-1})] + [f(1/2^{n-1}) - f(1/2^{n-2})] + [f(1/2^{n-2}) - f(1/2^{n-3})] + \cdots + [f(1/2) - f(0)] < n(-1/2) = -n/2 \to -\infty$ as $n \to \infty$. Since f is an increasing function, it follows that $f(x) \to -\infty$ as $x \to 0^+$. □

Limit at ∞

Claim. $\lim_{x\to\infty} f(x) = \infty$

The proof is similar, depending on the fact claim that $f(2^n) - f(2^{n-1}) > 1/2$ for all $n \in \mathbb{Z}^+$.

With this information, we can draw a very good sketch of the graph of f and can start looking at the algebraic properties of f.

Algebraic Properties of f

The key properties of logarithmic functions are that the log of a product is the sum of the logs, the log of a quotient is the difference of logs, and the log of something to a power is the power times the log. We can show that f has essentially the same properties.

Lemma 3. *Let* $x, y > 0$*,* $r \in \mathbb{Q}$*.*

(1) $f(xy) = f(x) + f(y)$
(2) $f(x/y) = f(x) - f(y)$
(3) $f(x^r) = rf(x)$

Both the second and third parts are consequences of the first. The first part can be proven by defining a new function $g(x) = f(xy)$ for fixed y and showing that $g'(x) = 1/x = f'(x)$, so that $f(xy)$ and $f(x)$ must differ by a constant. Writing $f(xy) = f(x) + c$ and letting $x = 1$, we find $c = f(y)$ and the first part follows.

Since f is continuous, $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$, it follows that $f : \mathbb{R}^+ \to \mathbb{R}$ is onto. In particular, f must take on the value 1 somewhere. Since f is $1-1$, we may define e to be the unique number such that $f(e) = 1$. It turns out that $f(x) = \log_e x$, that is, f is a logarithmic function to the base e. It can also be shown that $2 < x < 3$.

Claim. *If* $x > 0$*, then* $f(x) = \log_e x$*.*

We will actually prove only that if $\log_e x$ exists, then $f(x) = \log_e x$.

Proof. Let $x > 0$ have a logarithm y to base e, so $y = \log_e x$ and $e^y = x$. Then $f(x) = f(e^y) = yf(e) = y = \log_e x$. □

Note: In the preceding argument, y *had to be a rational number.*

We can now eliminate all pretense and rename f to be the *Natural Logarithm Function*, generally denoted by ln.

Properties of the Natural Logarithm Function

(1) $\ln x = \int_1^x 1/t\,dt$ for $x > 0$
(2) $\ln : \mathbb{R}^+ \to \mathbb{R}$
(3) ln is $1-1$ and onto.
(4) $\dfrac{d}{dx}(\ln x) = 1/x$ and ln is increasing.

(5) $\int 1/x\,dx = \ln|x|$
(6) $\dfrac{d^2 \ln x}{dx^2} = -1/x^2$ and the graph of ln is concave down.
(7) $\ln(xy) = \ln x + \ln y$
(8) $\ln(x/y) = \ln x - \ln y$
(9) $\ln(x^r) = r \ln x$
(10) $\ln(e) = 1$
(11) $\ln = \log_e$
(12) $\ln x = \begin{cases} < 0 \text{ for } 0 < x < 1 \\ = 0 \text{ for } x = 1 \\ > 0 \text{ for } x > 1 \end{cases}$

Logarithmic Di�erentiation

The properties of logarithms come in handy when calculating derivatives, particularly when the function being di�erentiated has variables in exponents.

The Method:

- Assume you have a function $f(x)$.
  Write $y = f(x)$.
- Take logs of both sides:
  $\ln y = \ln f(x)$
- Use the properties of logarithms simplify $\ln f(x)$.
- Di�erentiate implicitly.

Example of Logarithmic Di�erentiation

Suppose we wish to find the derivative of $(\sin x)^{2x+1}$.

- Write $y = (\sin x)^{2x+1}$
- Take logs of both sides to get $\ln y = \ln[(\sin x)^{2x+1}]$
- Use the properties of logs to get $\ln y = (2x+1)\ln\sin x$
- Di�erentiate implictly: $\dfrac{d}{dx}(\ln y) = \dfrac{d}{dx}((2x+1)\ln\sin x)$

$$\frac{(\frac{dy}{dx})}{y} = (2x+1)\cdot\frac{\cos x}{\sin x} + (\ln\sin x)\cdot 2$$
$$\frac{dy}{dx} = y[(2x+1)\cot x + 2\ln\sin x]$$
$$\frac{dy}{dx} = (\sin x)^{2x+1}[(2x+1)\cot x + 2\ln\sin x]$$

Inverse Functions

Consider a function $f : A \to B$. For each element $a \in A$ there is some element $b \in B$ such that $f(a) = b$.

Question: Given an arbitrary element $b \in B$, is there always a unique element $a \in A$ such that $f(a) = b$?

For the answer to be yes, two conditions must hold:

(1) For each element $b \in B$, there must be some $a \in A$ for which $f(a) = b$. In other words, B must actually be the range of f. We sometimes write $f(A) = B$ and say that the function $f : A \to B$ is *onto*.

(2) The element $a \in A$ such that $f(a) = b$ must be unique. In other words, there cannot be two distinct elements, $a_1, a_2 \in A$ with $a_1 \neq a_2$, such that $f(a_1) = f(a_2)$. Such a function is said to be *one-to-one* or $1 - 1$.

If $f : A \to B$ is $1 - 1$ and onto, then we can define a function $f^{-1} : B \to A$ by defining $f^{-1}(b)$ to be the unique $a \in A$ such that $f(a) = b$. f^{-1} is called the *inverse* of f.

Properties of Inverse Functions

- $f : A \to B$, $f^{-1} : B \to A$
- $f(a) = b$ if and only if $f^{-1}(b) = a$
- For each $a \in A$, $f^{-1} \circ f(a) = f^{-1}(f(a)) = a$. In other words, $f^{-1} \circ f$ is the identity function on A.
- For each $b \in B$, $f \circ f^{-1}(b) = f(f^{-1}(b)) = b$. In other words, $f \circ f^{-1}$ is the identity function on B.

Examples

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$.

f is $1-1$ and onto and has inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(x) = \frac{x}{2}$.

Question: How does one find an inverse?

Solution:

(1) Write down the formula $y = f(x)$ for the original function.

(2) Treat it as an equation and solve for x in terms of y. This gives a formula $x = f^{-1}(y)$.

(3) (Optional) If you want, interchange x and y to write the formula for the inverse in the form $y = f^{-1}(x)$.

Important: If one is using a notation using independent and dependent variables, things can get very confusing.

Question: What if $f : A \to B$ is $1 - 1$ but not onto?

Theoretical Answer: Define a new function $g : A \to f(A)$ by letting $g(x) = f(x) \forall x \in A$. g will be $1 - 1$ and onto and hence invertible.

Practical Answer: Pretend B is really $f(A)$.

Properties of Inverse Functions

- If an invertible function is continuous and is defined on an interval, then its inverse is continuous.
- If a is in the range of an invertible function f and $f'(f^{-1}(a)) \neq 0$, then f^{-1} is differentiable at a and $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$.

  This can be thought of as $\dfrac{dx}{dy} = \dfrac{1}{\left(\frac{dy}{dx}\right)}$.

  Alternate Notation: $(f^{-1})' = \dfrac{1}{f' \circ f^{-1}}$.

The Exponential Function

Since ln is $1-1$ and onto, it has an inverse function.

Definition 2 (The Exponential Function)**.** $\exp = \ln^{-1}$

Lemma 4. *If* $x \in \mathbb{Q}$*, then* $\exp(x) = e^x$.

Proof. Since $\exp = \ln^{-1}$, it follows that
$\ln(\exp(x)) = x$.

By the properties of the ln function, $\ln(e^x) = x\ln(e) = x$.

Since ln is $1-1$, the conclusion follows. □

Definition of Irrational Exponents

More generally, for any $x \in \mathbb{Q}$ and $a > 0$, suppose we let $y = a^x$, using the classical definition of an exponent. Then $\ln y = \ln(a^x) = x\ln(a)$. But then $y = \exp(\ln y) = \exp(x \ln a)$, so we must have $a^x = \exp(x \ln a)$. Since the expression on the right is defined for all $x \in \mathbb{R}$, $a > 0$, it's natural to use this for a general definition of an exponential.

Definition 3. *For* $a > 0$*,* $x \in \mathbb{R}$*,* $a^x = \exp(x \ln a)$.

As a special case, we have $e^x = \exp(x \ln e) = \exp(x)$.

We may thus write:

- $e^x = \exp(x)$
- $a^x = \exp(x \ln a) = e^{x \ln a}$

Claim 1. $\dfrac{de^x}{dx} = e^x$

Proof. Let $y = e^x$. Then $\ln y = x$. Differentiating, we get $y'/y = 1$,so $y' = y = e^x$. □

Using the fact that $\exp = \ln^{-1}$ and the properties of the ln function, one can show that exp has the properties of an exponential function. We can then summarize.

Properties of the Exponential Function

- $\exp = \ln^{-1}$
- $\exp : \mathbb{R} \to \mathbb{R}^+$
- exp is $1-1$ and onto.
- $\dfrac{d}{dx}(e^x) = \dfrac{d}{dx}(\exp x) = e^x$, $\int e^x\,dx = e^x$
- exp is increasing and concave up.
- $\exp(0) = 1$, $\exp(1) = e$.
- $e^x e^y = e^{x+y}$
- $e^x/e^y = e^{x-y}$
- $(e^x)^y = e^{xy}$

Exponential Growth and Decay

The exponential function satisfies the di�erential equation $y' = y$. We may ask whether this is the only such function. Obviously, it's not, since any constant multiple of the exponential function satisfies the same di�erential equation, so we modify the question to whether any other family of functions satisfies that di�erential equation. More generally, we obtain the following result.

Exponential Growth and Decay

Theorem. *If $f'(x) = kf(x)$ for some $k \in \mathbb{R}$ on some interval, then $f(x) = ae^{kx}$ for some $a \in \mathbb{R}$ on that interval.*

Proof. Note: There is a hole in this proof. See whether you can find it. Even better, see whether you can fix it

Suppose $f'(x) = kf(x)$ on some interval. Dividing both sides by $f(x)$, we get $\dfrac{f'(x)}{f(x)} = k$. Since the left hand side is the derivative of $\ln|f(x)|$, it follows that $\ln|f(x)| = kx + c$ for some $c \in \mathbb{R}$.

Exponentiating both sides, we get $|f(x)| = e^{kx+c} = c'e^{kx}$, where $c' = e^c$.

Letting $a = \begin{cases} c' \text{ if } f'(x) > 0 \\ -c' \text{ if } f'(x) < 0, \end{cases}$, we have $f(x) = ae^{kx}$. □

This theorem e�ectively shows that every function which changes at a rate proportional to its size must be an exponential function.

Examples

- Continuous Interest
- Radioactive Decay
- Population Growth

Mathematically, each of these situations is the same, with only the terminology being di�erent. In most cases, the independent variable represents time and is denoted by t, so we have functions of the form

$y = ae^{bt}$.

We generally have to find a and b before we can do anything else and we usually use known values of y, sometimes given subtly, in order to find a and b.

Newton's Law of Cooling

Modeling Newton's Law of Cooling leads to another di�erential equation whose solution involves exponential functions.

Newton's Law of Cooling is an empirical law which says the rate at which an object changes temperature is proportional to the di�erence between the temperature of the object and the ambient temperature.

To start, let's determine the relevant variables. Let:

- T be the temperature of the object,
- T_a be the ambient temperature,
- t be time, and let
- T_0 be the initial temperature of the object.

Newton's Law of Cooling

Since *the derivative measures rate of change*, Newton's Law of Cooling implies $\frac{dT}{dt} \propto T - T_0$. This may be written $\frac{dT}{dt} = b(T - T_0)$, where $b \in \mathbb{R}$ is the constant of proportionality.

To solve this separable di�erential equation, divide both sides by $T - T_0$ to get $\frac{1}{T - T_0}\frac{dT}{dt} = b$.

Integrate to get:

$\int \frac{1}{T - T_0}\, dT = \int b\, dt$.

$\ln |T - T_0| = bt + k$

$|T - T_0| = e^{bt+k} = e^{bt}e^{k} = ce^{bt}$, where $c = e^{k}$

Newton's Law of Cooling

$|T - T_0| = e^{bt+k} = e^{bt}e^{k} = ce^{bt}$, where $c = e^{k}$

Since $T - T_0$ is both di�erentiable and therefore continuous, and ce^{bt} is never 0, it follows from the *Intermediate Value Theorem* that either $T - T_0$ is always positive or always negative. If we let $a = \begin{cases} c & \text{if } T - T_0 > 0 \\ -c & \text{if } T - T_0 < 0 \end{cases}$, we can write

$T - T_0 = ae^{bt}$, which we may then solve for T to get

$T = T_0 + ae^{bt}$.

Separable Di�erential Equations

A differential equation is essentially an equation involving derivatives. Differential equations arise naturally in many applications and it is important to be able to solve them.

When we integrate $\int f(x)\,dx$, we are essentially solving the differential equation $\dfrac{dy}{dx} = f(x)$.

A slightly more general class of differential equations which are solvable in a similar manner is the class of *separable differential equations*.

Definition 4 (Separable Differential Equation)**.** *A separable differential equation is one which can be rewritten in the form* $g(y)y' = h(x)$*.*

Solutions of Separable Differential Equations

The general solution of a separable differential equation of the form $g(y)y' = h(x)$ is given by $\int g(y)\,dy = \int h(x)\,dx$.

Why?

- $g(y)y' = h(x)$
- $\int g(y)y'\,dx = \int h(x)\,dx$
- Using the *Substitution Technique*, and substituting $y = f(x)$, the left side reduces to $\int g(y)\,dy$.

Additional Evidence

If the argument given isn't sufficiently convincing, one may simply observe that if one takes a function $y = f(x)$ which satisfies the equation $\int g(y)\,dy = \int h(x)\,dx$ and differentiates implicitly, one gets $g(y)y' = h(x)$, which shows the function $y = f(x)$ satisfies the differential equation.

Inverse Trigonometric Functions

Let's start by looking at the sin function. Technically, since sin is not $1-1$, it does not have an inverse. We get around this problem with a technicality.

We define a new function, called the principal sine function and denoted by Sin, by restricting the domain. If one starts at 0, one sees the sin function starts repeating values once $\pi/2$ is reached. In order to get values of sin which are negative, one needs to go to the left of 0. As one goes left, one again starts duplicating values when $-\pi/2$ is reached.

Definition 5 (Principal Sine Function)**.** *Sin* $x : [-\pi/2, \pi/2] \to [-1, 1]$ *is defined by Sin* $x = \sin x$*.*

Definition of arcsin

Since the Principal Sine function is $1-1$, it has an inverse.

Definition 6 (Arcsin Function)**.** arcsin = Sin^{-1}.

We can think of $\arcsin x$ as the angle between $-\pi/2$ and $\pi/2$ whose sin is x.

Derivative of arcsin

We can use formulas obtained for derivatives of inverse functions to get a formula for the derivative of arcsin, but it's easier and better practice to use implicit differentiation as follows.

Let $y = \arcsin x$. We know that $x = \sin y$. Differentiating implicitly, we get $\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$, $1 = \cos y \frac{dy}{dx}$, $\frac{dy}{dx} = \frac{1}{\cos y}$.

We know $\sin y = x$, so if we use the basic trigonometric identity $\cos^2 y + \sin^2 y = 1$, we get $\cos^2 y + x^2 = 1$, so $\cos^2 y = 1 - x^2$, $\cos y = \pm\sqrt{1 - x^2}$.

However, since $y = \arcsin x$ is in the interval $[-\pi/2, \pi/2]$, it follows that $\cos y$ must be positive, so $\cos y = \sqrt{1 - x^2}$.

We conclude $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$, so $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}$.

Definition of arccos

We can define arccos in a manner similar to the way arcsin was defined. The natural interval to define the *Principal Cosine* function is $[0, \pi]$. So we define

Definition 7. $\text{Cos} : [0, \pi] \to [-1, 1]$ *by* $\text{Cos}\, x = \cos x$.

We naturally define arccos = Cos^{-1}.

It follows that $y = \arccos x$ if and only if $x = \text{Cos}\, y$, so we may think of $\arccos x$ as the angle between 0 and π whose cosine is x.

Derivative of arccos

Let $y = \arccos x$. It follows that $x = \cos y$. We may differentiate:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\cos y)$$

$$1 = -\sin y \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

Since $\cos y = x$, we may write $\cos^2 y + \sin^2 y = 1$, $x^2 + \sin^2 y = 1$, $\sin^2 y = 1 - x^2$, $\sin y = \pm\sqrt{1 - x^2}$.

Since $y = \arccos x$, it follows that $y \in [0, \pi]$, so $\sin y \geq 0$ and $\sin y = +\sqrt{1 - x^2}$.

It follows that $\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$.

We thus have the formula $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$.

arcsin and arccos are Complementary

From the formulas for the derivatives of arcsin and arccos, we have $\frac{d}{dx}(\arccos x + \arcsin x) = \frac{d}{dx}(\arccos x) + \frac{d}{dx}(\arcsin x) = -\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = 0$.

Since only constant functions have a derivative of 0, it follows that $\arccos x + \arcsin x = k$ for some constant k.

To find k, we may plug in any angle for which we know arcsin and arccos. The simplest choice is 0, giving

$\arccos 0 + \arcsin 0 = k$

$\pi/2 + 0 = k$

$k = \pi/2$

$\arccos x + \arcsin x = \pi/2$

In other words, arcsin and arccos are complementary angles. This is obviously true when both are acute angles, in which case they are both angles of the same right triangle, but it's also true if they are obtuse or even negative!

Definition of Arctangent

Definition 8 (Principal Tangent). $\mathrm{Tan} : [-\pi/2, \pi/2] \to \mathbb{R}$ *is defined by* $\mathrm{Tan}\, x = \tan x$.

Definition 9 (Arctangent). $\arctan = \mathrm{Tan}^{-1}$.

We can think of $\arctan x$ as the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x.

Derivative of Arctangent

Let $y = \arctan x$. Then $x = \tan y$.

$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$

$1 = \sec^2 y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\sec^2 y}$

Since $1 + \tan^2 y = \sec^2 y$, $1 + x^2 = \sec^2 y$, so $\frac{dy}{dx} = \frac{1}{1+x^2}$.

We thus get the formula

$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$.

Definition of Arcsecant

Definition 10 (Principal Secant). *Sec* : $[0, \pi/2) \cup (\pi/2, \pi] \to (-\infty, -1] \cup [1, \infty)$ *is defined by* $Sec x = \sec x$.

Definition 11 (Arcsecant). $\text{arcsec} = Sec^{-1}$.

We may think of $\text{arcsec}\, x$ as the angle between 0 and π whose secant is x.

Derivative of Arcsecant

Let $y = \text{arcsec}\, x$.
Then $x = \sec y$.
$\frac{d}{dx}(x) = \frac{d}{dx}(\sec y)$.
$1 = \sec y \tan y \frac{dy}{dx}$.
$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$.

Since $1 + \tan^2 y = \sec^2 y$ and $\sec y = x$, it follows that $\tan^2 y = \sec^2 y - 1$, $\tan^2 y = x^2 - 1$, $\tan y = \pm\sqrt{x^2 - 1}$.

It follows that $\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2 - 1}}$.

Since $\sec y$ and $\tan y$ always have the same sign, it follows that $\frac{dy}{dx}$ is always positive, so $\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2 - 1}}$.

Derivative of Arcsecant

We thus have the formula $\frac{d}{dx}(\text{arcsec}\, x) = \frac{1}{|x|\sqrt{x^2 - 1}}$.

Integration By Parts

Integration by Parts is a technique that enables us to calculate integrals of functions which are derivatives of products. Its genesis can be seen by di�erentiating a product and then fiddling around.

- Write out the formula for the derivative of a product $f(x)g(x)$.
  $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$
- Treat the formula as an equation and solve for $f(x)g'(x)$.
  $f(x)g'(x) = \frac{d}{dx}(f(x)g(x)) - f'(x)g(x)$

Integration By Parts

- Find a formula for the integral of $f(x)g'(x)$ by integrating the formula for $f(x)g'(x)$.

  $\int f(x)g'(x)\,dx = \int \frac{d}{dx}(f(x)g(x))\,dx - \int f'(x)g(x)\,dx$
- Simplifying, we get the *Integration by Parts* formula:

  $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

Alternate Notation

Letting $u = f(x)$ and $v = g(x)$, so $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, we can write the Integration by Parts formula in either of the forms

$\int uv'\,dx = uv - \int u'v\,dx$

$\int u\,dv = uv - \int v\,du$

Example: Calculating $\int x^2 \ln x\,dx$

Let $f(x) = \ln x$, $g'(x) = x^2$. Then $f'(x) = \frac{1}{x}$, $g(x) = \frac{x^3}{3}$.

Plugging that into the Integration by Parts formula, we obtain

$$\begin{aligned}\int x^2 \ln x\,dx &= (\ln x)\cdot\frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx \\ &= \frac{x^3\ln x}{3} - \frac{1}{3}\int x^2\,dx \\ &= \frac{x^3\ln x}{3} - \frac{x^3}{9}.\end{aligned}$$

Integration By Parts – Determining $f(x)$ and $g'(x)$

Ideally, $f(x)$ will be a function which is easy to differentiate and whose derivative is simpler than $f(x)$ itself, while $g'(x)$ is a function that's easy to integrate, since if we can't find $g(x)$ it will be impossible to continue with Integration By Parts.

Trigonometric functions such as $\sin x$, $\cos x$ and $\sec^2 x$ are good candidates for $g'(x)$, as are exponential functions.

Logarithmic functions are good candidates for $f(x)$, since they are difficult to integrate but easy to differentiate and their derivatives do not involve logarithms.

Integrating Powers of Trigonometric Functions

Integrals of the form $\int \sin^m x \cos^n x\,dx$ can always be calculated when m and n are positive integers.

The techniques used also sometimes, but not always, work when the exponents are not positive integers.

Integrals involving other trigonometric functions can always, if necessary, be written in terms of sin and cos.

Odd Powers

The simplest case is if either sin or cos occurs to an odd power in an integrand. In this case, substitute for the other. We can do this even the other doesn't occur!

After making the substitution and simplifying, the trigonometric function that occurred to an odd power may still occur to an even power, but we can make use of the basic identity $\cos^2 x+\sin^2 x = 1$ to eliminate its presence.

Example – $\int \cos^3 x\, dx$

Here, cos occurs to an odd power, so we substitute $u = \sin x$, even though $\sin x$ doesn't appear in the integrand.

Continuing, we get $\dfrac{du}{dx} = \cos x$, $du = \cos x\, dx$, $dx = \dfrac{du}{\cos x}$.

We can now substitute into the integral to get $\int \cos^3 x\, dx = \int \cos^3 x \cdot \dfrac{du}{\cos x} = \int \cos^2 x\, du$.

cos still appears, but to an even power, so we make use of the basic trigonometric identity to write:

$$\int \cos^2 x\, du = \int (1 - \sin^2 x)\, dx = \int (1 - u^2)\, du.$$

The rest of the calculation is routine:

$$\int (1 - u^2)\, du = u - \frac{u^3}{3} = \sin x - \frac{\sin^3 x}{3}.$$

We conclude $\int \cos^3 x\, dx = \sin x - \dfrac{\sin^3 x}{3} + k$.

Even Powers of sin and cos

When we have only even powers of sin and cos, the substitution for one of them doesn't work. In this case, there are two alternatives.

- Use Integration By Parts or, equivalently, Reduction Formulas.
- Use Double Angle Formulas

We will pursue the latter alternative.

Review of Double Angle Formulas

The formulas about the values of trigonometric functions at sums and differences of angles come in handy. The seminal formula is the formula for the cosine of a difference: $\cos(u - v) = \cos u \cos v + \sin u \sin v$.

This formula may be derived by drawing a unit circle in standard position (with center at the origin) along with central angles u, v

and $u - v$ terminating in the points $P(\cos u, \sin u)$, $Q(\cos v, \sin v)$ and $R(\cos(u - v), \sin(u - v))$.

If one notes the chord joining P and Q has the same length as the chord joining R and $(1, 0)$, uses the distance formula to observe the consequence

$$(\cos u - \cos v)^2 + (\sin u - \sin v)^2 = (\cos(u - v) - 1)^2 + (\sin(u - v) - 0)^2,$$

and simplifies, one obtains the formula

$$\cos(u - v) = \cos u \cos v + \sin u \sin v.$$

Cosine of a Sum

The key here is the observation $u + v = u - (-v)$. We take the formula $\cos(u - v) = \cos u \cos v + \sin u \sin v$ and replace v by $-v$ as follows:

$$\cos(u + v) = \cos(u - (-v)) = \cos u \cos(-v) + \sin u \sin(-v).$$

Using the identities $\cos(-v) = \cos v$ and $\sin(-v) = -\sin v$, we get

$$\cos(u + v) = \cos u \cos v + \sin u(-\sin v) = \cos u \cos v - \sin u \sin v.$$

This is the basis of the formulas we need for integration, but we will review the formulas for the sin of a sum or difference as well.

The Sine of a Sum

The key observation here is that the sine of an angle is equal to the cosine of its complement. We thus calculate

$$\sin(u+v) = \cos(\pi/2-[u+v]) = \cos([\pi/2-u]-v) = \cos(\pi/2-u)\cos v + \sin(\pi/2 - u)\sin v = \sin u \cos v + \cos u \sin v.$$

The Sine of a Difference

We can get this from the sine of a sum by recognizing $u-v = u+(-v)$ and calculating as follows:

$$\sin(u - v) = \sin(u + (-v)) = \sin u \cos(-v) + \cos u \sin(-v).$$

Again using the identities $\cos(-v) = \cos v$ and $\sin(-v) = -\sin v$, we get:

$$\sin(u - v) = \sin u \cos v + (\cos u)(-\sin v) = \sin u \cos v - \cos u \sin v.$$

Summary of the Formulas

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$
$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$
$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$
$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$

The Double Angle Formulas

It's easy to use the formulas for sums to get *double angle formulas* for sin and cos, by observing $2u = u + u$:

$$\cos(2u) = \cos(u+u) = \cos u \cos u - \sin u \sin u = \cos^2 u - \sin^2 u$$

$$\sin(2u) = \sin(u+u) = \sin u \cos u + \cos u \sin u = 2 \sin u \cos u$$

The Double Angle Formulas We Use

The double angle formula for cosine has two variations which we obtain using the basic trigonometric identity:

$$\cos 2u = \cos^2 u - \sin^2 u = \cos^2 u - (1 - \cos^2 u) = \cos^2 u - 1 + \cos^2 u = 2\cos^2 u - 1$$

$$\cos 2u = \cos^2 u - \sin^2 u = (1 - \sin^2 u) - \sin^2 u = 1 - 2\sin^2 u$$

We don't actually use these formulas directly in integration, but take them and solve one for $\cos^2 u$ and the other for $\sin^2 u$.

$\cos^2 u$

We take the formula $\cos 2u = 2\cos^2 u - 1$ and solve for $\cos^2 u$ as follows:

$$\cos 2u = 2\cos^2 u - 1,$$
$$2\cos^2 u = 1 + \cos 2u,$$
$$\cos^2 u = \frac{1 + \cos 2u}{2}.$$

$\sin^2 u$

We take the formula $\cos 2u = 1 - 2\sin^2 u$ and solve for $\sin^2 u$ as follows:

$$\cos 2u = 1 - 2\sin^2 u,$$
$$2\sin^2 u = 1 - \cos 2u,$$
$$\sin^2 u = \frac{1 - \cos 2u}{2}.$$

Integrating Even Powers of Sine and Cosine

To integrate even powers, we simply write any even power as a power of a square and replace $\cos^2 x$ by $\frac{1 + \cos 2x}{2}$ and replace $\sin^2 x$ by $\frac{1 - \cos 2x}{2}$. This effectively reduces the power, although we wind up with more terms in the integrand.

We may have to repeat this process many times, so the integration gets extremely messy.

Example: $\int \cos^2 x \, dx$

We calculate $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \int \frac{1}{2} + \frac{1}{2}\cdot\cos 2x\,dx = \frac{1}{2}\cdot x + \frac{1}{2}\cdot\frac{\sin 2x}{2} = \frac{x}{2} + \frac{\sin 2x}{4} + c$.

Note we needed to make a substitution $u = 2x$, or a guess, to integrate the second term.

Example: $\int \sin^2 x \cos^2 x$

We calculate $\int \sin^2 x\cos^2 x\,dx = \int \frac{1-\cos 2x}{2}\cdot\frac{1+\cos 2x}{2}\,dx = \frac{1}{4}\int 1 - \cos^2 2x\,dx = \frac{1}{4}\int 1 - \frac{1+\cos 4x}{2}\,dx = \frac{1}{4}\int 1 - \frac{1}{2} - \frac{\cos 4x}{2}\,dx = \frac{1}{4}\int \frac{1}{2} - \frac{\cos 4x}{2}\,dx = \frac{1}{8}\int 1 - \cos 4x\,dx = \frac{1}{8}(x - \frac{\sin 4x}{4}) + c$.

Obviously, the calculations can get very messy very quickly.

Trigonometric Substitutions

Integrals involving sums and di�erences of squares can often be calculated using *trigonometric substitutions*. These are technically substitutions involving inverse trigonometric functions, such as $\theta = \arcsin x$ or $\theta = \arctan x$, but these explicit substitutions don't need to be written down.

The key to trigonometric substitutions is the *Pythagorean Theorem*:

If the legs of a right triangle have lengths a and b and the hypotenuse has length c, then $a^2 + b^2 = c^2$.

This can also be written as $c^2 - a^2 = b^2$ or $c^2 - b^2 = a^2$.

The way we choose a substitution depends on whether the integrand contains a sum or a di�erence of squares.

Sum of Squares

If the integrand contains a sum of squares, such as $a^2 + b^2$, then we consider a triangle with legs a and b and hypotenuse $\sqrt{a^2+b^2}$. We may call one of the acute angles θ. For the sake of definiteness, assume a is adjacent to θ and b is opposite θ, although this may be reversed. Also, assume a is constant, while b, and thus $\sqrt{a^2+b^2}$ as well, includes the variable of integration.

We will undoubtedly need $\sqrt{a^2+b^2}$, so we observe $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$, so $\sqrt{a^2+b^2} = \frac{a}{\cos\theta}$.

We may need b, in which case we observe $\tan\theta = \frac{b}{a}$, so $b = a\tan\theta$.

Sum of Squares

Assuming the variable of integration is x, we will need dx. If we di□erentiate $\frac{d}{d\theta}(b) = \frac{d}{d\theta}(a\tan\theta) = a\sec^2\theta$ and then multiply by $d\theta$ to get $\frac{d}{d\theta}(b)\cdot d\theta = a\sec^2\theta\cdot d\theta$, we will be able to solve for dx.

Di□erence of Squares

If the integrand contains a di□erence of squares, such as $c^2 - a^2$, then we consider a triangle with hypotenuse c and legs a and $\sqrt{c^2 - a^2}$. We may call one of the acute angles θ. For the sake of definiteness, assume a is adjacent to θ and $\sqrt{c^2 - a^2}$ is opposite θ, although this may be reversed.

We will undoubtedly need $\sqrt{c^2 - a^2}$.

If a is constant, we observe $\tan\theta = \frac{\sqrt{c^2 - a^2}}{a}$, so $\sqrt{c^2 - a^2} = a\tan\theta$.

If c is constant, observe $\sin\theta = \frac{\sqrt{c^2 - a^2}}{c}$, so $\sqrt{c^2 - a^2} = a\sin\theta$.

Notice the idea: Combine the side that is needed with the constant side.

Di□erence of Squares

If a involves the variable of integration, we note $\cos\theta = \frac{a}{c}$, so $a = c\cos\theta$. To find dx, we'd then di□erentiate this.

skippause If c involves the variable of integration, we still note $\cos\theta = \frac{a}{c}$, but then sove $c = \frac{a}{\cos\theta}$. We again di□erentiate to find dx.

Then What?

Once we've solved for all the sides of the triangle and for dx, we substitute for each in the original integrand.

If we're lucky, and this will happen most of the time unless we've missed seeing something obvious, we will wind up with an integral we can evaluate.

Once we've evaluated that integral, which gives us a result in terms of θ, we look at the triangle and rewrite the result in terms of the original variable.

Example: $\int \frac{1}{x^2 + 4}\,dx$

Note: This can be evaluated by guessing, or using the substitution $u = x/2$, motivated by trying to get the denominator in the form $(\textit{something})^2 + 1$, but illustrates the use of a trigonometric substitution.

Since $x^2 + 4 = x^2 + 2^2$, we let the legs be x and 2 and the hypotenuse $\sqrt{x^2+4}$. We let one acute angle be θ and let x be opposite θ and 2 adjacent to θ. These choices for the legs may be reversed and everything will still work.

Example: $\displaystyle\int \frac{1}{x^2+4}\,dx$

We need $x^2 + 4$, so first we find $\sqrt{x^2+4}$. Since $\sqrt{x^2+4}$ is the hypotenuse and the constant side 2 is the adjacent side, we use the cosine function, observing $\cos\theta = \dfrac{2}{\sqrt{x^2+4}}$, so $\sqrt{x^2+4} = \dfrac{2}{\cos\theta}$.

We actually need $x^2 + 4$. Since $x^2 + 4 = \left(\sqrt{x^2+4}\right)^2$, we get $x^2 + 4 = \left(\dfrac{2}{\cos\theta}\right)^2 = \left(\dfrac{4}{\cos^2\theta}\right)$.

Example: $\displaystyle\int \frac{1}{x^2+4}\,dx$

We don't need x for itself, but we need it to find dx. Since x is the opposite side and the constant side 2 is the adjacent side, we use the tangent function and observe $\tan\theta = \dfrac{x}{2}$, so $x = 2\tan\theta$.

We can now di erentiate to get $\dfrac{dx}{d\theta} = 2\sec^2\theta$ and then multiply both sides by $d\theta$ to get $dx = 2\sec^2\theta d\theta$.

We can now substitute into the original integral to get $\displaystyle\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4/\cos^2\theta}\cdot 2\sec^2\theta\,d\theta = \int \frac{\cos^2\theta}{4}\cdot 2\sec^2\theta\,d\theta = \frac{1}{2}\int 1\,d\theta = \frac{1}{2}\theta$.

Example: $\displaystyle\int \frac{1}{x^2+4}\,dx$

We can now look at the triangle, observe that θ is the angle whose tangent is $x/2$, so that $\theta = \arctan(x/2)$, and conclude $\displaystyle\int \frac{1}{x^2+4}\,dx = \frac{1}{2}\arctan(x/2) + c$.

$$\text{Example: } \int \frac{1}{\sqrt{x^2-1}}\,dx$$

Here we have a di□erence of squares. We draw a triangle and let x be the hypotenuse and let 1 and $\sqrt{x^2-1}$ be the legs. Call one of the acute angles θ, let 1 be the leg adjacent to θ and $\sqrt{x^2-1}$ the leg opposite θ.

We need $\sqrt{x^2-1}$, which is the opposite side. Since the constant 1 is the adjacent side, we use the tangent and note $\tan\theta = \frac{\sqrt{x^2-1}}{1}$, so $\sqrt{x^2-1} = \tan\theta$.

To get dx, we start by getting x. Since x is the hypotenuse and the constant leg 1 is the adjacent side, we use the cosine function. We observe $\cos\theta = \frac{1}{x}$, so $x = \frac{1}{\cos\theta} = \sec\theta$. Di□erentiating, we get $\frac{dx}{d\theta} = \sec\theta\tan\theta$, so $dx = \sec\theta\tan\theta d\theta$.

$$\text{Example: } \int \frac{1}{\sqrt{x^2-1}}\,dx$$

We can now substitute back into the integral: $\int \frac{1}{\sqrt{x^2-1}}\,dx = \int \frac{1}{\tan\theta}\cdot\sec\theta\tan\theta d\theta = \int \sec\theta\,d\theta$.

We have previously found $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$, so we get $\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|\sec\theta + \tan\theta|$.

We now look at the triangle and note $\sec\theta = \frac{x}{1} = x$, while $\tan\theta = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$, so $\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x + \sqrt{x^2-1}| + c$.

$$\text{Example: } \int \frac{1}{\sqrt{5+4x-x^2}}\,dx$$

This is a little more complicate than other examples. At first glance, it may not seem to involve either a sum or di□erence of squares, but completing the squares demonstrates otherwise. This just goes to show the most di□cult aspects of calculus involve algebra.

It's probably easier to look at the additive inverse of the quadratic inside the radical, so consider x^2-4x-5. Since $(x-2)^2 = x^2-4x+4$, we have $x^2-4x-5 = (x^2-4x+4)-4-5 = (x-2)^2-9$. So we may write $5 + 4x - x^2 = 9 - (x-2)^2$ and write the integral as $\int \frac{1}{\sqrt{9-(x-2)^2}}\,dx$.

Example: $\int \frac{1}{\sqrt{5+4x-x^2}}\,dx$

We thus draw a triangle where the hypotenuse is 3 and the legs are $x - 2$ and $\sqrt{9-(x-2)^2}$. Let θ be one of the acute angles. We'll let $x - 2$ be the opposite leg and $\sqrt{9-(x-2)^2}$ be the adjacent leg.

We need $\sqrt{9-(x-2)^2}$. Since that's the adjacent leg and the constant side 3 is the hypotenuse, we use the cosine function and write $\cos\theta = \frac{\sqrt{9-(x-2)^2}}{3}$, so $\sqrt{9-(x-2)^2} = 3\cos\theta$.

Example: $\int \frac{1}{\sqrt{5+4x-x^2}}\,dx$

We don't need x for itself, but we need dx. First we find $x-2$. Since that's the opposite side and the constant side 3 is the hypotenuse, we use the sine function and write $\sin\theta = \frac{x-2}{3}$, so $x - 2 = 3\sin\theta$.

To find dx, we differentiate implicitly: $\frac{d}{d\theta}(x-2) = \frac{d}{d\theta}(3\sin\theta)$, so $\frac{dx}{d\theta} = 3\cos\theta$ and $dx = 3\cos\theta d\theta$.

Example: $\int \frac{1}{\sqrt{5+4x-x^2}}\,dx$

We're now ready to substitute in the integral: $\int \frac{1}{\sqrt{9-(x-2)^2}}\,dx = \int \frac{1}{3\cos\theta}\,3\cos\theta\,d\theta = \int 1\,d\theta = \theta$.

Looking at the triangle, we observe θ is an angle whose sine is $\frac{x-2}{3}$, so $\theta = \arcsin\left(\frac{x-2}{3}\right)$ and so we conclude $\int \frac{1}{\sqrt{5+4x-x^2}}\,dx = \arcsin\left(\frac{x-2}{3}\right) + c$.

Integration of Rational Functions

A rational function is a quotient of polynomials. Through the use of the algebraic technique of *partial fraction decomposition*, it is theoretically possible to rewrite any rational function as a sum of terms which may be integrated using techniques we've studied.

The algebraic theorem behind partial fraction decomposition requires that the numerator of the rational function have lower degree than the denominator. This is not a significant problem, since through the use of long division any rational function can be written as a polynomial, which is easily integrated, plus a rational function where the degree of the numerator is smaller than the degree of the denominator.

From now on, we will assume that our rational functions have numerators of lower degree than their denominators.

Partial Fraction Decomposition

The partial fraction decomposition also depends on rewriting the denominator as a product of powers of distinct first and second degree polynomials. It also requires the second degree polynomials be unfactorable. In theory, every polynomial can be so written.

According to the partial fraction decomposition, once a rational function has been so written, it can be expressed as a sum of terms where the denominators of the terms are the individual factors of the denominators of the rational function raised to integer powers up to the power each appears in the original rational function and the numerators are one degree lower than the polynomials in the denominators.

What does this mean?

The Meaning of Partial Fractions Decomposition

If the denominator contains a linear factor occurring to given power, the partial fractions decomposition will contain terms with the same linear factor, raised to every integer power up to the power it occurs in the original denominator. The numerator of each term will be a constant.

In other words, if the denominator contains a factor $(ax+b)^p$, the partial fractions decomposition will contain terms $\frac{\alpha_1}{ax+b} + \frac{\alpha_2}{(ax+b)^2} + \frac{\alpha_3}{(ax+b)^3} + \cdots + \frac{\alpha_p}{(ax+b)^p}$.

The Meaning of Partial Fractions Decomposition

If the denominator contains a quadratic factor occurring to given power, the partial fractions decomposition will contain terms with the same quadratic factor, raised to every integer power up to the power it occurs in the original denominator. The numerator of each term will be a linear function.

In other words, if the denominator contains a factor $(ax^2 + bx + c)^p$, the partial fractions decomposition will contain terms $\frac{\alpha_1 x + \beta_1}{ax^2 + bx + c} + \frac{\alpha_2 x + \beta_2}{(ax^2 + bx + c)^2} + \frac{\alpha_3 x + \beta_3}{(ax^2 + bx + c)^3} + \cdots + \frac{\alpha_p x + \beta_p}{(ax^2 + bx + c)^p}$.

Examples

If the original denominator contains a factor $(5x + 3)^4$, the partial fractions decomposition will contain terms $\frac{a}{5x + 3} + \frac{b}{(5x + 3)^2} + \frac{c}{(5x + 3)^3} + \frac{d}{(5x + 3)^4}$.

If the original denominator contains a factor $(x^2 + 2x + 3)^3$, the partial fractions decomposition will contain terms $\frac{ax + b}{x^2 + 2x + 3} + \frac{cx + d}{(x^2 + 2x + 3)^2} + \frac{ex + f}{(x^2 + 2x + 3)^3}$.

Computing the Constants

Once we know the form of the partial fractions decomposition, we still have to find the constants involved. These can be determined in at least two di□erent ways. Often, a combination of the two ways is the most e□cient.

With either method, the first step is to rewrite the expression in the partial fractions decomposition by getting a common denominator, which will be the same as the original denominator, and adding the numerators. This numerator must be equal to the original denominator for all values of the independent variable. We may write $L = R$, where L represents the numerator of the original rational function and R represents the numerator we get after adding the terms of the partial fractions decomposition together. For convenience, we will refer to the independent variable as x.

Calculating the Constants

One method for calculating the constants is to choose values for x which make individual terms of R 0. This will often enable us to quickly evaluate at least some of the constants in R.

A second method is to equate the individual coe□cients of L and R. This will give a system of linear equations which can be solved to find all the constants of R.

The first method will work when all the factors in the original denominator are linear and all occur to the first power.

One optimal strategy is to get all the information one can using the first method, and then equate some coe□cients to get the rest of the constants.

$$\text{Example: } \frac{x+23}{x^2+x-20}$$

$x^2+x-20=(x-4)(x+5)$, so we may write $\frac{x+23}{x^2+x-20}=\frac{a}{x-4}+\frac{b}{x+5}$.

Getting a common denominator: $\frac{a}{x-4}+\frac{b}{x+5}=\frac{a}{x-4}\frac{x+5}{x+5}+\frac{b}{x+5}\frac{x-4}{x-4}=\frac{a(x+5)+b(x-4)}{(x+5)(x-4)}$.

Equating numerators, we know $a(x+5)+b(x-4)=x+23$.

Finding the Constants Using the First Method

We know $a(x+5)+b(x-4)=x+23$.

Since $x-4=0$ when $x=4$, we plug $x=4$ into the equation and get $a(4+5)+b(4-4)=4+23$, $9a=27$, $a=3$.

Since $x+5=0$ when $x=-5$, we plug $x=-5$ into the equation to get $a(-5+5)+b(-5-4)=-5+23$, $-9b=18$, $b=-2$.

We conclude $\frac{x+23}{x^2+x-20}=\frac{3}{x-4}+\frac{-2}{x+5}$, or $\frac{x+23}{x^2+x-20}=\frac{3}{x-4}-\frac{2}{x+5}$.

Finding the Constants Using the Second Method

We know $a(x+5)+b(x-4)=x+23$.

Multiplying out the numerator from the partial fractions expansion and combining like terms, we get $a(x+5)+b(x-4)=ax+5a+bx-4b=(a+b)x+(5a-4b)$. We may thus write $(a+b)x+(5a-4b)=x+23$.

Equating coe□cients, we get

$a+b=1$
$5a-4b=23$.

Finding the Constants Using the Second Method

$a + b = 1$
$5a - 4b = 23$

We can solve this many ways. For example, we can solve for a in the first equation $a = 1 - b$ and plug it into the second equation: $5(1 - b) - 4b = 23$. We can then solve $5 - 5b - 4b = 23$, $5 - 9b = 23$, $-9b = 18$, $b = -2$.

We can now use the fact $a = 1 - b$ to get $a = 1 - (-2) = 3$.

We have again found $\dfrac{x + 23}{x^2 + x - 20} = \dfrac{3}{x - 4} - \dfrac{2}{x + 5}$

Remember: We still have to calculate the integral!

The Actual Integration

After finding partial fractions expansion, we still have to carry out the integration.

Some terms will have linear denominators raised to powers. These are easily integrated using a substitution of the form $u =$ (the linear factor).

For example, to integrate $\displaystyle\int \frac{12}{(5x + 3)^4}, dx$, the substitution $u = 5x + 3$ will work quickly.

Terms with quadratic denominators raised to powers are somewhat more involved, but may be integrated using trigonometric substitutions.

Quadratic Denominators

The key is that an unfactorable quadratic can always be written as a sum of squares using the method of *completing the square*. Once the denominator is written as a sum of squares, perhaps raised to a power, a trigonometric substitution may be used.

Example: $\displaystyle\int \frac{2x - 5}{(x^2 - 6x + 25)^3}\, dx$.

We may start by completing the square on $x^2 - 6x + 25$ as follows:

Since half of -6 is -3, we note $(x - 3)^2 = x^2 - 6x + 9$, so $x^2 - 6x = (x - 3)^2 - 9$ and $x^2 - 6x + 25 = [(x - 3)^2 - 9] + 25 = (x - 3)^2 + 16$.

We can now rewrite the integral in the form $\displaystyle\int \frac{2x - 5}{((x - 3)^2 + 16)^3}\, dx$.

$$\int \frac{2x - 5}{((x - 3)^2 + 16)^3}\, dx$$

We may draw a right triangle with acute angle θ, hypotenuse $\sqrt{x^2 - 6x + 25} = \sqrt{(x-3)^2 + 16}$, and legs $x - 3$ and 4. Let $x - 3$ be the opposite side and 4 be the adjacent side.

To find $\sqrt{(x-3)^2 + 16}$, since that is the hypotenuse and the constant leg 4 is the adjacent side, we use the cosine. We write $\cos\theta = \frac{4}{\sqrt{(x-3)^2 + 16}}$, so $\sqrt{(x-3)^2 + 16} = 4\sec\theta$.

To find x, we note $x - 3$ is the opposite side, so we use the tangent. We write $\tan\theta = \frac{x-3}{4}$, so $x - 3 = 4\tan\theta$ and $x = 3 + 4\tan\theta$.

We can now find dx using di�erentiating: $\frac{dx}{d\theta} = 4\sec^2\theta$, $dx = 4\sec^2\theta d\theta$.

$$\int \frac{2x-5}{((x-3)^2+16)^3}\, dx$$

Using:
$\sqrt{(x-3)^2 + 16} = 4\sec\theta$
$x = 3 + 4\tan\theta$
$dx = 4\sec^2\theta d\theta$,
we can substitute in the original integral:

$$\int \frac{2x-5}{((x-3)^2+16)^3}\, dx = \int \frac{2\cdot(3+4\tan\theta)-5}{(4\sec\theta)^6}\cdot 4\sec^2\theta d\theta = \frac{1}{4^5}\int \frac{8\tan\theta+1}{\sec^4\theta}\, d\theta = \frac{1}{4^5}\int (8\tan\theta+1)\cos^4\theta\, d\theta = \frac{1}{4^5}\int 8\sin\theta\cos^3\theta + \cos^4\theta\, d\theta.$$

Both terms involve powers of sine and cosine, which can be integrated using standard methods.

General Integration Strategy

Based on the methods learned in class, which do not constitute a complete set of methods, we can use the following strategy to calculate indefinite integrals.

This strategy, or a variation personalized by the student, will enable a student to integrate most of the integrals run across in elementary calculus for which the calculation of an indefinite integral is feasible.

General Integration Strategy

Start by integrating term-by-term. For each individual term, use the following strategy:

- Check whether the integral can be evaluated immediately, either because the integrand is the derivative of an elementary function of an algebraic or trigonometic manipulation can put it in a form which is.

- Look for a relatively straightforward substitution.
- Look to see whether the integrand fits into one of the special situations studied, including:
  - A product of powers of trigonometric functions
  - A sum or di□erence of squares
  - A rational function
- Try Integration By Parts
- As a last resort, try integration by parts with $g'(x) = 1$

Indeterminate Forms and L'Hôpital's Rule

Limits involving indeterminates of the form $\lim_{x\to c} \frac{f(x)}{g(x)}$ can often be calculated using a convenient theorem known as L'Hôpital's Rule. There are really approximately sixty di□erent cases of L'Hôpital's rule, but they are all variations of the following.

Theorem 5 (L'Hôpital's Rule). *If* $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ *and* $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$, *then* $\lim_{x\to c} \frac{f(x)}{g(x)} = L$.

About L'Hôpital's rule

- The text includes additional hypotheses, but these are implied by the requirement $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$.
- The rule is stated for ordinary limits as $x \to c$ for some real number c, but the conclusion also holds for one-sided limits and for limits at ∞ and $-\infty$ also hold.
- The rule is stated for the numerator and denominator both $\to 0$, but the conclusion also holds if both approach either $\pm\infty$. This includes the possibility that one $\to \infty$ and the other $\to -\infty$.
- The rule is stated for a finite limit, but the conclusion also holds for infinite limits.

Indeterminates Other Than Quotients

The indeterminates L'Hôpital's Rule deals may be thought of symbolically as the $\frac{0}{0}$ and $\frac{\infty}{\infty}$ cases. There are other types of indeterminates to which L'Hôpital's Rule doesn't directly apply but which can be transformed so that L'Hôpital's Rule can be made use of indirectly.

These cases may be thought of symbolically as the following cases.

$0 \cdot \infty$

∞^0

0^0

1^∞

Indeterminate Products

If we have a product, we may transform it into a quotient by dividing one factor by the reciprocal of the other. Symbolically, we may think of it as rewriting $0 \cdot \infty$ as either $\frac{0}{1/\infty}$ or $\frac{\infty}{1/0}$. The former becomes the $\frac{0}{0}$ case and the latter becomes the $\frac{\infty}{\infty}$ case.

Indeterminate Exponentials

If we have one of the exponential indeterminates, we may use the definition $a^b = e^{b \ln a}$. If we can calculate the limit of $b \ln a$, we can use that to find the limit of a^b. If the limit of $b \ln a$ is L, then the limit of $a^b = e^{b \ln a}$ will be e^L.

Since the exponential function is continuous everywhere, this follows from the theorem about continuous functions that if a function f is continuous at L and $\lim_{x\to c} g(x) = L$, then $\lim_{x\to c} f \circ g(x) = f(L)$.

Symbolically, we write:

$\infty^0 = e^{0 \ln \infty}$, $0^0 = e^{0 \ln 0^+}$ and $1^\infty = e^{\infty \cdot \ln 1}$.

In each case, we are left with the $0 \cdot \infty$ case in the exponent.

Improper Integrals

We have four different basic types of improper integrals, two with limits of integration at either ∞ or $-\infty$ and two with discontinuities at a finite limit of integration.

(1) $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$
(2) $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$
(3) If $a < b$ and f is not continuous at b, $\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$
(4) If $a < b$ and f is not continuous at a, $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$

We can often use the definition to evaluate improper integrals.

Convergence, Divergence and Notation

If an improper integral has a numerical value, we say it converges; if an improper integral does not converge, we say it diverges.

Suppose $f(x) \geq 0$.

- If $\int_a^\infty f(x)\,dx$ converges, we write $\int_a^\infty f(x)\,dx < \infty$.
- If $\int_a^\infty f(x)\,dx$ diverges, we write $\int_a^\infty f(x)\,dx = \infty$.

We use an analogous notation for other types of improper integrals with non-negative integrands.

Variations

Some integrals have problems at both limits of integration or inside their intervals of integration.

We deal with that by splitting the integral into a sum of integrals, each of with has a problem at just one endpoint.

For such an integral to converge, each of the individual integrals needs to converge.

The P-Test

Improper integrals where the integrand is of the form $\frac{1}{x^p}$ turn out to be particularly important. It turns out the case $p = 1$ is a dividing line between integrals which converge and integrals which diverge.

Consider $\int_1^\infty \frac{1}{x^p}\,dx$.

By definition, $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx$. We'll consider the case $p = 1$ separately.

The P-Test

For $p = 1$, we get $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln t = \infty$.

For $p \neq 1$, we get $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^t =$ $\lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p} = \begin{cases} \infty & \text{if } p < 1 \\ \frac{1}{p-1} & \text{if } p > 1 \end{cases}$

We thus get $\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} = \infty & \text{if } p \leq 1 \\ < \infty & \text{if } p > 1 \end{cases}$

Integral from 0

Consider $\int_0^1 \frac{1}{x^p}\,dx$. Once again, $p = 1$ is a dividing line.

By definition, $\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \int_1^t \frac{1}{x^p}\,dx$. We'll consider the case $p = 1$ separately.

For $p = 1$, we get $\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx = -\lim_{t\to 0^+} \int_1^t \frac{1}{x}\,dx =$ $-\lim_{t\to 0^+} \ln t = \infty$.

For $p \neq 1$, we get $\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-p}\,dx = \lim_{t\to 0^+} \frac{x^{1-p}}{1-p}\Big|_t^1 =$ $\lim_{t\to 0^+} \frac{1}{1-p} - \frac{t^{1-p}}{1-p} = \begin{cases} \frac{1}{1-p} & \text{if } p < 1\infty \quad \text{if } p > 1 \end{cases}$

We thus get $\int_0^1 \frac{1}{x^p}\,dx \begin{cases} < \infty & \text{if } p < 1 \\ = \infty & \text{if } p \geq 1 \end{cases}$

The P-Tests Generalized

The arguments used to develop the P-Tests can be used in more general settings to give four di�erent variations.

Let $a < \alpha$. $\int_\alpha^\infty \frac{1}{(x-a)^p}\,dx \begin{cases} = \infty & \text{if } p \leq 1 \\ < \infty & \text{if } p > 1 \end{cases}$

Let $\beta < b$. $\int_{-\infty}^\beta \frac{1}{(b-x)^p}\,dx \begin{cases} = \infty & \text{if } p \leq 1 \\ < \infty & \text{if } p > 1 \end{cases}$

Let $a < b$. $\int_a^b \frac{1}{(x-a)^p}\,dx \begin{cases} < \infty & \text{if } p < 1 \\ = \infty & \text{if } p \geq 1 \end{cases}$

Let $a < b$. $\int_a^b \frac{1}{(b-x)^p}\,dx \begin{cases} < \infty & \text{if } p < 1 \\ = \infty & \text{if } p \geq 1 \end{cases}$

These can be considered prototypes that are used in conjunction with the *Comparison Test*.

The Comparison Tests

The Comparison Tests (there are several related tests) essentially state that smaller functions are more likely to have integrals which converge.

The Comparison Tests are used to determine whether improper integrals converge or diverge *without having to actually calculate the integrals themselves*.

The most basic Comparison Test is the following.

Theorem 6 (Comparison Test). *Let* $0 \leq f(x) \leq g(x)$ *for* $x \geq a$.

(1) *If* $\int_a^\infty g(x)\,dx < \infty$, *then* $\int_a^\infty f(x)\,dx < \infty$.
(2) *If* $\int_a^\infty f(x)\,dx = \infty$, *then* $\int_a^\infty g(x)\,dx = \infty$.

It is really only necessary that there be some $\alpha \in \mathbb{R}$ such that $0 \leq f(x) \leq g(x)$ for $x \geq \alpha$.

Variations of the Comparison Test

Suppose $0 \leq f(x) \leq g(x)$ for $x \geq \alpha$ for some $\alpha \in \mathbb{R}$. Then:

(1) If $\int_a^\infty g(x)\,dx < \infty$, then $\int_a^\infty f(x)\,dx < \infty$.
(2) If $\int_a^\infty f(x)\,dx = \infty$, then $\int_a^\infty g(x)\,dx = \infty$.

Suppose $0 \leq f(x) \leq g(x)$ for $x \leq \beta$ for some $\beta \in \mathbb{R}$. Then:

(1) If $\int_{-\infty}^b g(x)\,dx < \infty$, then $\int_{-\infty}^b f(x)\,dx < \infty$.

(2) If $\int_{-\infty}^{b} f(x)\,dx = \infty$, then $\int_{-\infty}^{b} g(x)\,dx = \infty$.

Variations of the Comparison Test

Suppose $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ are improper integrals at a and $0 \le f(x) \le g(x)$ for $a < x < \beta$ for some $\beta \le b$. Then:

(1) If $\int_a^b g(x)\,dx < \infty$, then $\int_a^b f(x)\,dx < \infty$.
(2) If $\int_a^b f(x)\,dx = \infty$, then $\int_a^b g(x)\,dx = \infty$.

Suppose $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ are improper integrals at b and $0 \le f(x) \le g(x)$ for $\alpha < x < b$ for some $\alpha \ge a$. Then:

(1) If $\int_a^b g(x)\,dx < \infty$, then $\int_a^b f(x)\,dx < \infty$.
(2) If $\int_a^b f(x)\,dx = \infty$, then $\int_a^b g(x)\,dx = \infty$.

Using the Comparison Test

We'll consider integrals of the form $\int_a^{\infty} f(x)\,dx$, where $f(x) > 0$. Other types of improper integrals are analyzed similarly. The Comparison Test is generally used as follows.

One starts by finding some function $g(x) > 0$ which is similar in size to $f(x)$ but whose convergence is easier to analyze. Functions of the form $g(x) = \dfrac{1}{x^p}$ are among the most frequent candidates, since the P-Test can be used.

Showing Convergence

Suppose after deciding on $g(x)$ we observe $\int_a^{\infty} g(x)\,dx < \infty$.

One then expects that $\int_a^{\infty} f(x)\,dx < \infty$.

If we're lucky, $f(x) \le g(x)$ and we can immediately apply the Comparison Test to prove $\int_a^{\infty} f(x)\,dx < \infty$.

If $f(x)$ is not smaller than $g(x)$, we need to find another function $g^*(x)$ with $f(x) < g^*(x)$ for which $\int_a^{\infty} g^*(x)\,dx < \infty$. If we can find such a function, we can use the Comparison Test to prove $\int_a^{\infty} f(x)\,dx < \infty$.

Showing Divergence

Suppose, after deciding on $g(x)$, we observe $\int_a^{\infty} g(x)\,dx = \infty$.

One then expects that $\int_a^{\infty} f(x)\,dx = \infty$.

We we're lucky, $f(x) \ge g(x)$ and we can immediately apply the Comparison Test to prove $\int_a^{\infty} f(x)\,dx = \infty$.

If $f(x)$ is smaller than $g(x)$, we need to find a function $g^*(x)$ with $f(x) \ge g^*(x)$ for which $\int_a^{\infty} g^*(x)\,dx = \infty$.

We can then use the Comparison Test to prove $\int_a^{\infty} f(x)\,dx = \infty$.

Trapezoid Rule

The Trapezoid Rule is used to estimate an integral $\int_a^b f(x)\,dx$.
Let:
$h = \Delta x = \dfrac{b-a}{n}$
$x_k = a + kh$
$y_k = f(x_k)$
$\int_a^b f(x)\,dx \approx \dfrac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$
$\int_a^b f(x)\,dx \approx \dfrac{b-a}{2n}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$

Area Under a Parabola

It will be shown that the integral of a quadratic function depends only on the width of the interval over which it's integrated and the values of the function at the midpoint and endpoints.

To simplify the calculations, assume that the interval is of the form $[-h, h]$ and that the quadratic function is of the form $f(x) = ax^2 + bx + c$. $\int_{-h}^h f(x)\,dx$ may be integrated easily using the Fundamental Theorem of Calculus.

$$\begin{aligned}
\int_{-h}^{h} f(x)\,dx &= \int_{-h}^{h} ax^2 + bx + c\,dx \\
&= ax^3/3 + bx^2/2 + cx\big|_{-h}^{h} \\
&= ah^3/3 + bh^2/2 + ch \\
&- \{a(-h)^3/3 + b(-h)^2/2 + c(-h)\} \\
&= ah^3/3 + bh^2/2 + ch + ah^3/3 - bh^2/2 + ch \\
&= 2ah^3/3 + 2ch \\
&= \frac{h}{3} \cdot (2ah^2 + 6c)
\end{aligned}$$

Let

$$\begin{aligned}
y_{-h} &= f(-h) = ah^2 - bh + c \\
y_0 &= f(0) = c \\
y_h &= f(h) = ah^2 + bh + c
\end{aligned}$$

Since $y_{-h} + y_h = 2ah^2 + 2c$, it is easily seen that $2ah^2 + 6c = y_{-h} + 4y_0 + y_h$, and thus $I = \dfrac{h}{3} \cdot (y_{-h} + 4y_0 + y_h)$.

Simpson's Rule

Simpson's Rule may be used to approximate $\int_a^b f(x)\,dx$. It takes the idea of the Trapezoid Rule one level higher.

Rationale

Partition the interval $[a, b]$ evenly into n subintervals, where n is even, so that each subinterval has width $h = \dfrac{b-a}{n}$ and let $y_k = f(x_k)$.

Estimate the integral over adjacent pairs of integrals by the integral of a quadratic function agreeing with f at the midpoint and endpoints of the interval.

Simpson's Rule

$\int_{x_0}^{x_2} f(x)\,dx \approx \dfrac{h}{3} \cdot (y_0 + 4y_1 + y_2)$

$\int_{x_2}^{x_4} f(x)\,dx \approx \dfrac{h}{3} \cdot (y_2 + 4y_3 + y_4)$

$\int_{x_4}^{x_6} f(x)\,dx \approx \dfrac{h}{3} \cdot (y_4 + 4y_5 + y_6)$

...

$\int_{x_{n-2}}^{x_n} f(x)\,dx \approx \dfrac{h}{3} \cdot (y_{n-2} + 4y_{n-1} + y_n)$

If everything is added together, we obtain the estimate

$\int_a^b f(x)\,dx \approx \dfrac{h}{3} \cdot (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$.

This is known as Simpson's Rule.

Midpoint Rule

$\int_a^b f(x)\,dx \approx h \cdot \left(f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right)$

Trapezoid Rule

$$\begin{aligned} &\int_a^b f(x)\,dx \\ \approx\; &\frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) \\ =\; &\frac{b-a}{2n}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) \end{aligned}$$

Simpson's Rule

$$\begin{aligned} &\int_a^b f(x)\,dx \\ \approx\; &\frac{h}{3} \cdot (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \\ =\; &\frac{b-a}{3n} \cdot (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \end{aligned}$$

Error Estimates

Let E_T be the error in the Trapezoid Rule.
Let E_M be the error in the Midpoint Rule.
Let E_S be the error in Simpson's Rule.

Let K be a bound on the second derivative.
Let K^* be a bound on the fourth derivative.
$|E_T| \le \dfrac{K(b-a)^3}{12n^2}$
$|E_M| \le \dfrac{K(b-a)^3}{24n^2}$
$|E_S| \le \dfrac{K^*(b-a)^5}{180n^4}$

Sequences

A sequence is essentially just a list.

Definition 12 (Sequence of Real Numbers). *A sequence of real numbers is a function* $\mathbb{Z} \cap (n, \infty) \to \mathbb{R}$ *for some real number* n.

Don't let the description of the domain confuse you; it's just a fancy way of saying the domain consists of a set of consecutive integers starting with some integer but never ending.

In most cases, the domain will be either the set of positive integers or the set of non-negative integers.

Try to recognize that the entire definition is just a fancy, but precise, way of saying a sequence is a list of numbers.

We can have sequences of objects other than real numbers, but in this course we will restrict ourselves to sequences of real numbers and will from now on just refer to sequences.

Notation

We generally use the notation $\{a_n\}$ to denote a sequence, just as we often use the notation $f(x)$ to denote a function.

n is the independent variable, but when studying sequences we refer to it as the index.

We'll often define a sequence by giving a formula for a_n, just as we often define an ordinary function $f(x)$ by giving a formula.

Example: $a_n = \dfrac{1}{n}$. This sequence can also be described by $1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots$.

Example: $b_n = n^2$. This sequence can also be described by $1, 4, 9, 16, 25, \ldots$
.

Convergence of a Sequence

We often want to know whether the terms of a sequence $\{a_n\}$ approach some limit as $n \to \infty$. This is analogous to an ordinary limit at infinity, so we define a limit of a sequence by appropriately modifying the definition of an ordinary limit at infinity. Recall:

Definition 13 (Limit at Infinity). $\lim_{x\to\infty} f(x) = L$ *if for every* $\ > 0$ *there is some real number* N *such that* $|f(x) - L| < \ $ *whenever* $x > N$.

We get a definition of a limit of a sequence by replacing $f(x)$ by a_n and replacing x by n, obtaining:

Definition 14 (Limit of a Sequence). $\lim a_n = L$ *if for every* $\ > 0$ *there is some real number* N *such that* $|a_n - L| < \ $ *whenever* $n > N$.

Convergence of a Sequence

Note: We may write $\lim_{n\to\infty} a_n$, but it is acceptible to simply write $\lim a_n$ since there is no reasonable interpretation other than for $n \to \infty$.

If a sequence has a limit, we say it converges; otherwise, we say it diverges.

Properties of Limits of Sequences

Limits of sequences share many properties with ordinary limits. Each of the following properties may be proven essentially the same way the analogous properties are proven for ordinary limits. (Each of these properties depends on the limit on the right side existing.)

- $\lim(a_n \pm b_n) = \lim a_n \pm \lim b_n$
- $\lim ka_n = k \lim a_n$
- $\lim k = k$
- $\lim a_n b_n = \lim a_n \lim b_n$
- If $\lim b_n \neq 0$, $\lim \dfrac{a_n}{b_n} = \dfrac{\lim a_n}{\lim b_n}$

Sequences Through Ordinary Functions

The similarity of the definitions of limits of sequences and limits at infinity yield the following corollary:

Theorem 7. *Consider a sequence* $\{a_n\}$ *and an ordinary function* f. *If* $a_n = f(n)$ *and* $\lim_{x\to\infty} f(x) = L$, *then* $\lim a_n = L$.

Proof. Suppose the hypotheses are satisfied and let $\ > 0$. Since $\lim_{x\to\infty} f(x) = L$, if follows there must be some $N \in \mathbb{R}$ such that $|f(x) - L| < \ $ whenever $x > N$. Since $a_n = f(n)$, it follows that $|a_n - L| < \ $ whenever $n > N$. □

Applying the Analogy

This theorem implies each of the following limits, which can also be proven independently.

- $\lim \frac{1}{n} = 0$
- $\lim \sqrt[n]{n} = 1$
- $\lim (1 + 1/n)^n = e$
- $\lim \frac{\ln n}{n} = 0$
- $\lim \frac{n}{2^n} = 0$

A similarly flavored limit which needs to be proven separately is $\lim \frac{2^n}{n!} = 0$.

Using L'Hôpital's Rule

L'Hôpital's Rule cannot be used directly to find limits of sequences, but it can be used indirectly.

We can often find $\lim a_n$ by finding a function $f(x)$ such that $a_n = f(n)$ and then using L'Hôpital's Rule to find $\lim_{x\to\infty} f(x)$.

Example

We want to find $\lim \frac{n \ln n}{n^2 + 1}$. We let $f(x) = \frac{x \ln x}{x^2 + 1}$. We can then use L'Hôpital's Rule to find

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x \ln x}{x^2 + 1} = \lim_{x\to\infty} \frac{x \cdot \frac{1}{x} + (\ln x) \cdot 1}{2x}$$
$$= \lim_{x\to\infty} \frac{1 + \ln x}{2x} = \lim_{x\to\infty} \frac{1/x}{2} = 0,$$

so $\lim \frac{n \ln n}{n^2 + 1} = 0$.

Monotonic Sequences

Sometimes it is possible and even necessary to determine whether a sequence converges without having to find what it converges to. This is often the case with monotonic sequences.

Definition 15 (Increasing). *A sequence* $\{a_n\}$ *is increasing if* $a_k \leq a_{k+1}$ *for all* k *in its domain.*

Definition 16 (Strictly Increasing). *A sequence* $\{a_n\}$ *is strictly increasing if* $a_k < a_{k+1}$ *for all* k *in its domain.*

Definition 17 (Decreasing). *A sequence* $\{a_n\}$ *is decreasing if* $a_k \geq a_{k+1}$ *for all* k *in its domain.*

Definition 18 (Strictly Decreasing). *A sequence* $\{a_n\}$ *is strictly decreasing if* $a_k > a_{k+1}$ *for all* k *in its domain.*

Monotonic Sequences

Definition 19 (Monotonicity). *If a sequence is either increasing or decreasing, it is said to be monotonic.*

Definition 20 (Boundedness). *A sequence* $\{a_n\}$ *is said to be bounded if there is a number* $B \in \mathbb{R}$ *such that* $|a_n| \leq B$ *for all* n *in the domain of the sequence.* B *is referred to as a bound.*

Theorem 8 (Monotone Convergence Theorem). *A monotonic sequence converges if and only if it is bounded.*

The Monotone Convergence Theorem becomes very important in determining the convergence of infinite series.

The Completeness Axiom

The proof of the Monotone Convergence Theorem depends on:

The Completeness Axiom: *If a nonempty set has a lower bound, it has a greatest lower bound; if a nonempty set has an upper bound, it has a least upper bound.*

The terms lower bound, greatest lower bound, upper bound and least upper bound mean precisely what they sound like.

Exercise: Write down precise definitions.

We will give a proof of the Monotone Convergence Theorem for an increasing sequence. A similar proof can be created for a decreasing sequence.

Proof of the Monotone Convergence Theorem

Proof. If a sequence is increasing and has a limit, it is clearly bounded below by its first term and bounded above by its limit and thus must be bounded, so we'll just show that a sequence which is increasing and bounded must have a limit.

So suppose $\{a_n\}$ is increasing and bounded. It must have an upper bound and thus, by the Completeness Axiom, must have a least upper bound B. Let $\quad > 0$. There must be some element a_N of the sequence such that $B - \quad < a_N \leq B$. $a_N \leq B$ since B is an upper bound. If $B - \quad < a_N$ was false for all elements of the sequence, $B - \quad$ would be an upper bound smaller than B, so B would not be the least upper bound.

It follows that if $n > N$, $B - \quad < a_N < a_n \leq B$, so $|a_n - B| < \quad$ and it follows from the definition of a limit that $\lim a_n = B$. □

Infinite Series

Definition 21 (Infinite Series). *An expression* $a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$ *is called an infinite series.*

The terms of a series form a sequence, but in a series we attempt to *add* them together rather than simply list them.

We don't actually have to start with $k = 1$; we could start with any integer value although we will almost always start with either $k = 1$ or $k = 0$.

Convergence of Infinite Series

We want to assign some meaning to a *sum* for an infinite series. It's naturally to add the terms one-by-one, e□ectively getting a sum for part of the series. This is called a partial sum.

Definition 22 (Partial Sum). $S_n = \sum_{k=1}^{n} a_k$ *is called the* n^{th} *partial sum of the series* $\sum_{k=1}^{\infty} a_k$.

If the sequence $\{S_n\}$ of a series converges to some number S, we say the series converges to S and write $\sum_{k=1}^{\infty} a_k = S$. We call S the sum of the series.

If the series doesn't converge, we say it diverges.

The Series 0.33333...

With the definition of a series, we are able to give a meaning to a non-terminating decimal such as 0.33333... by viewing it as $0.3 + 0.03 + 0.003 + 0.0003 + \cdots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots = \sum_{k=1}^{\infty} \frac{3}{10^k}$.

Using the definition of convergence and a little algebra, we can show this series converges to $\frac{1}{3}$ as follows.

The Series 0.33333...

The n^{th} partial sum $S_n = \sum_{k=1}^{n} \frac{3}{10^k} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^{n-1}}$.

Multiplying both sides by 10, we get $10S_n = \sum_{k=1}^{n} \frac{3}{10^{k-1}} = 3 + \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^{n-2}}$.

Subtracting, we get $10S_n - S_n = 3 - \frac{3}{10^{n-1}}$, so $9S_n = 3 - \frac{3}{10^{n-1}}$ and $S_n = \frac{1}{3} - \frac{1/3}{10^{n-1}}$.

Clearly, $\lim S_n = \frac{1}{3}$, so the series $\sum_{k=1}^{\infty} \frac{3}{10^k}$ converges to $\frac{1}{3}$.

Geometric Series

A similar analysis may be applied to any geometric series.

Definition 23 (Geometric Series). *A geometric series is a series which may be written in the form* $\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \dots$.

In other words, $a_k = ar^{k-1}$. The first term is generally referred to as a and r is called the common ratio.

We can obtain a compact formula for the partial sums as follows:

Geometric Series

Letting $S_n = a + ar + ar^2 + ar^3 + \dots ar^{n-1}$, we can multiply both sides by the common ratio r to get $rS_n = ar + ar^2 + ar^3 + \dots ar^{n-1} + ar^n$.

Subtracting, we get $S_n - rS_n = a - ar^n$

$(1 - r)S_n = a(1 - r^n)$

$S_n = \frac{a(1 - r^n)}{1 - r}$ if $r \neq 1$.

Geometric Series

$S_n = \frac{a(1 - r^n)}{1 - r}$ if $r \neq 1$.

If $|r| < 1$, it is clear that $r^n \to 0$ as $n \to \infty$, so $S_n \to \frac{a}{1 - r}$.

If $|r| > 1$, then $|r^n| \to \infty$ as $n \to \infty$, so $\{S_n\}$ clearly diverges.

If $r = -1$, then S_n oscillates back and forth between 0 and $2a$, so $\{S_n\}$ clearly diverges.

If $r = 1$, then $S_n = a + a + a + \dots + a = na$, so $\{S_n\}$ clearly diverges.

We may summarize this information by noting the geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ converges to $\frac{a}{1 - r}$ if $|r| < 1$ but diverges if $|r| \geq 1$.

Note on an Alternate Derivation

We could have found S_n di erently by noting the factorization $1 - r^n = (1-r)(1 + r + r^2 + \ldots r^{n-1})$, which is a special case of the general factorization formula $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots ab^{n-2} + b^{n-1})$.

It immediately follows that $1 + r + r^2 + \ldots r^{n-1} = \dfrac{1-r^n}{1-r}$.

Positive Term Series

Definition 24 (Positive Term Series). *A series* $\sum_{k=1}^{\infty} a_k$ *is called a* Positive Term Series *is* $a_k \geq 0$ *for all* k*.*

Theorem 9. *A positive term series converges if and only if its sequence of partial sums is bounded.*

Proof. Looking at the sequence of partial sums, $S_{n+1} = \sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1} = S_n + a_{n+1} \geq S_n$, since $a_{n+1} \geq 0$. Thus $\{S_n\}$ is monotonic and, by the Monotone Convergence Theorem, converges is and only if it's bounded. □

Note and Notation

This can be used to show a series converges but its more important purpose is to enable us to prove the Comparison Test for Series.

Notation: When dealing with positive term series, we may write $\sum_{k=1}^{\infty} a_k < \infty$ when the series converges and $\sum_{k=1}^{\infty} a_k = \infty$ when the series diverges.

This is analogous to the notation used for convergence of improper integrals with positive integrands.

Example: $\sum_{k=1}^{\infty} \frac{1}{k^2}$ Converges

Proof. Let $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Since, for $k \geq 2$, $\frac{1}{k^2} \leq \frac{1}{x^2}$ if $k - 1 \leq x \leq k$, it follows that $\frac{1}{k^2} = \int_{k-1}^{k} \frac{1}{k^2}\, dx \leq \int_{k-1}^{k} \frac{1}{x^2}\, dx$.

Thus $0 \leq S_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{n} \int_{k-1}^{k} \frac{1}{x^2}\,dx = 1 + \int_{1}^{n} \frac{1}{x^2}\,dx = 1 + \left[-\frac{1}{x}\right]_1^n = 1 + \left[-\frac{1}{n}\right] - (-1) = 2 - 1/n \leq 2.$

Since the sequence of partial sums is bounded, the series converges. □

Estimating the Error

Estimating $\sum_{k=1}^{\infty} \frac{1}{k^2}$ by $\sum_{k=1}^{n} \frac{1}{k^2}$ leaves an error $\sum_{k=n+1}^{\infty} \frac{1}{k^2}$.

Using the same type of reasoning used to show the series converges shows this sum is no greater than $\int_{n}^{\infty} \frac{1}{x^2}\,dx$, which can be evaluated as follows:

$$\int_{n}^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_{n}^{t} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_n^t$$

$$= \lim_{t\to\infty} \left[-\frac{1}{t}\right] - \left[-\frac{1}{n}\right] = \lim_{t\to\infty} \left[\frac{1}{n} - \frac{1}{t}\right] = \frac{1}{n}$$

We thus see estimating the series by the n^{th} partial sum leaves an error no larger than $\frac{1}{n}$, which can be made as small as desired by making n large enough.

The Comparison Test

Recall:

Theorem 10 (Comparison Test for Improper Integrals). *Let* $0 \leq f(x) \leq g(x)$ *for* $x \geq a$.

(1) *If* $\int_a^\infty g(x)\,dx < \infty$, *then* $\int_a^\infty f(x)\,dx < \infty$.
(2) *If* $\int_a^\infty f(x)\,dx = \infty$, *then* $\int_a^\infty g(x)\,dx = \infty$.

The Comparison Test

The Comparison Test for Improper Integrals has a natural analogue for Positive Term Series:

Theorem 11 (Comparison Test for Positive Term Series). *Let* $0 \leq a_n \leq b_n$ *for sufficiently large* n.

(1) *If* $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\sum_{n=1}^{\infty} a_n < \infty$.

(2) *If* $\sum_{n=1}^{\infty} a_n = \infty$, *then* $\sum_{n=1}^{\infty} b_n = \infty$.

The Comparison Test for Positive Term Series is used analogously to the way the Comparison Test for Improper Integrals is used.

Proof. Suppose the integral converges. For convenience, we will assume $\alpha = 1$. The proof can easily be modified if the integral is defined for some other α, but the argument is made most clearly without that complication.

Since $f(x)$ is clearly decreasing, $a_k \leq f(x)$ for $k-1 \leq x \leq k$, so $a_k \leq \int_{k-1}^{k} f(x)\,dx$ and $S_n = \sum_{k=1}^{n} a_k \leq a_1 + \int_1^n f(x)\,dx$.

Since the improper integral converges, the integral on the right is bounded. Thus the sequence of partial sums is bounded and the series must converge.

If the integral diverges, we may use the observation $S_n \geq \int_1^{n+1} f(x)\,dx$ to show the sequence of partial sums is not bounded and the series must diverge. □

Error Estimation

The proof of the Integral Test provides a clue about the error involved if one uses a partial sum to estimate the sum of an infinite series.

If one estimates the sum of a series $\sum_{k=1}^{\infty} a_k$ by its n^{th} partial sum $s_n = \sum_{k=1}^{n} a_k$, the error will equal the sum $\sum_{k=n+1}^{\infty} a_k$ of the terms not included in the partial sum.

If the series is a positive term series and $a_k = f(k)$ for a decreasing function $f(x)$, the analysis used in proving the Integral Test leads to the conclusion that this error is bounded by $\int_n^{\infty} f(x)\,dx$.

Example

Suppose we estimate the sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$ by s_{100} and want a bound on the error.

The error will be bounded by $\int_{100}^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_{100}^{t} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_{100}^{t}$
$= \lim_{t\to\infty} \left[-\frac{1}{t}\right] - \left[-\frac{1}{100}\right] = \frac{1}{100}$.

Determining a Number of Terms to Use

We can also figure out how many terms are needed to estimate a sum to within a predetermined tolerance . Using the same notation as before, this can be guaranteed if

(1) $$\int_n^\infty f(x)\,dx \le \ .$$

We can look at (1) as an inequality in n and solve for n.

This may be easier said than done.

Example

Suppose we want to estimate $\sum_{k=1}^{\infty} \frac{1}{k^3}$ to within 10^{-8}. We need to find n such that $\int_n^\infty \frac{1}{x^3}\,dx \le 10^{-8}$.

Integrating: $\int_n^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_n^t \frac{1}{x^3}\,dx = \lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_n^t$

$= \lim_{t\to\infty} \left[-\frac{1}{2t^2}\right] - \left[-\frac{1}{2n^2}\right] = \frac{1}{2n^2}$.

So we need $\frac{1}{2n^2} \le 10^{-8}$, which may be solved as follows:

$10^8 \le 2n^2$

$5 \cdot 10^7 \le n^2$

$\sqrt{5 \cdot 10^7} \le n$

Since $\sqrt{5 \cdot 10^7} \approx 7071.07$, we need to add 7072 terms to estimate the sum to within 10^{-8}.

Standard Series

P-Test for Series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} < \infty & \text{if } p > 1 \\ = \infty & \text{if } p \le 1. \end{cases}$$

Geometric Series

$$\sum_{k=0}^{\infty} ar^k \begin{cases} \text{converges} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \ge 1. \end{cases}$$

Absolute Convergence

Definition 25 (Absolute Convergence). $\sum_{k=1}^{\infty} a_k$ *is said to be absolutely convergent if* $\sum_{k=1}^{\infty} |a_k|$ *is convergent.*

Theorem 13. *A series which is absolutely convergent is convergent.*

Clearly, if this theorem wasn't true, the terminology of absolute convergence would be very misleading.

Proof of Absolute Convergence Theorem

Proof. Suppose $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

Let $a_k^+ = \begin{cases} a_k & \text{if } a_k \geq 0 \\ 0 & \text{if } a_k < 0. \end{cases}$

Let $a_k^- = \begin{cases} -a_k & \text{if } a_k \leq 0 \\ 0 & \text{if } a_k > 0. \end{cases}$

The terms of the positive term series $\sum_{k=1}^{\infty} a_k+$ and $\sum_{k=1}^{\infty} a_k-$ are both bounded by the terms of the convergent series $\sum_{k=1}^{\infty} |a_k|$. It follows immediately that $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(a_k^+ - a_k^-) = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-$ also converges. □

Definition 26 (Conditional Convergence). *A convergent series which is not absolutely convergent is said to be conditionally convergent.*

Testing for Absolute Convergence

All the tests devised for positive term series automatically double as tests for absolute convergence.

We will study one more test for convergence, the Ratio Test.

Ratio Test

The Ratio Test is useful for series which behave almost like geometric series but for which it can be di□cult to use the Comparison Test. It is not very useful for series that ordinarily would be compared to P-series.

The ratio test is usually stated as a test for absolute convergence, but can also be thought of as a test for convergence of positive term series. We state both versions below and use whichever version is more convenient.

Ratio Tests

Theorem 14 (Ratio Test for Positive Term Series). *Consider a positive term series* $\sum_{k=1}^{\infty} a_k$ *and let* $r = \lim_{k\to\infty} \dfrac{a_{k+1}}{a_k}$.

If $r < 1$*, then the series converges.*
If $r > 1$*, then the series diverges.*
If $r = 1$ *or the limit doesn't exist, the ratio test is inconclusive.*

Theorem 15 (Ratio Test for Absolute Convergence). *Consider a series* $\sum_{k=1}^{\infty} a_k$ *and let* $r = \lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right|$.

If $r < 1$*, then the series converges absolutely.*
If $r > 1$*, then the series diverges.*
If $r = 1$ *or the limit doesn't exist, the ratio test is inconclusive.*

Proof of the Ratio Test

We prove the ratio test for positive term series.

Proof. If $r > 1$, the terms of the series eventually keep getting larger, so the series clearly must diverge. Thus, we need only consider the case $r < 1$. Choose some $R \in \mathbb{R}$ such that $r < R < 1$. There must be some $N \in \mathbb{Z}$ such that $\frac{a_{k+1}}{a_k} < R$ whenever $k \geq N$. We thus have $a_{N+1} < a_N R$, $a_{N+2} < a_{N+1}R < a_N R^2$, $a_{N+3} < a_{N+2}R < a_N R^3$, and so on. Since $a_N + a_N R + a_N R^2 + a_N R^3 + \dots$ is a geometric series which common ratio $0 < R < 1$, it must converge. By the Comparison Test, the original series must converge as well. □

Strategy For Testing Convergence

It's important to have a strategy to determine whether a series $\sum_{k=1}^{\infty} a_k$ converges. The following is one reasonable strategy.

- Begin by making sure the individual terms converge to 0, since if the terms don't approach 0 then we know the series must diverge and there's no reason to check further.
- Next, check whether the series is one of the standard series, such as a P-Series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ or a Geometric Series $\sum_{k=1}^{\infty} ar^{k-1}$. If so, we can immediately determine whether it converges. Otherwise, we continue.
- We start by testing for absolute convergence.

Strategy For Testing Convergence

- Find a reasonable series to compare it to. One way is to look at the different terms and factors in the numerator and denominator, picking out the largest (using the general criteria powers of logs << powers << exponentials << factorials), and replacing anything smaller than the largest type by, as appropriate, 0 (for terms) or 1 (for factors). If we are lucky, we can

  use the resulting series along with the comparison test to determine whether our original series is absolutely convergent. If we're not lucky, we have to try something else.
- If the series seems to almost be geometric, the Ratio Test is likely to work.
- As a last resort, we can try the Integral Test.
- If the series is not absolutely convergent, we may be able to show it converges conditionally either by direct examination or by using the Alternating Series Test.

Strategy for Analyzing Improper Integrals

Essentially the same strategy may be used to analyze convergence of improper integrals.

Morphing Geometric Series Into Power Series

Suppose we take the geometric series $1 + r + r^2 + r^3 + \ldots$, which we know converges to $\frac{1}{1-r}$ for $|r| < 1$, and replace r by x:

$1 + x + x^2 + x^3 + \ldots$ converges to $\frac{1}{1-x}$ for $|x| < 1$.

We haven't really changed anything, but $1 + x + x^2 + x^3 + \ldots$ looks a little like a polynomial. It's an example of a *power series*.

Power Series

Definition 27 (Power Series). *An expression* $\sum_{n=0}^{\infty} a_n(x-c)^n$ *is called a power series centered at* c*.*

Note: When we write $\sum_{n=0}^{\infty} a_n(x-c)^n$*, we really mean* $a_0 + \sum_{n=1}^{\infty} a_n(x-c)^n$*, since* $a_0(x-c)^0$ *isn't defined when* $x = c$*, but it's more convenient to just write the sum starting from* $n = 0$*.*

Power Series

We will mostly consider power series centered at 0, written in the form $\sum_{n=0}^{\infty} a_n x^n$, but most of the facts about such series apply to series centered elsewhere.

Within their intervals of convergence, power series can be manipulated like polynomials. They can be added, subtracted and multiplied in the natural way and they can be differentiated and integrated term by term.

Radius of Convergence

Theorem 16. *Given a power series* $\sum_{n=0}^{\infty} a_n(x-c)^n$*, there is some* $R \geq 0$*, possibly* ∞*, such that the series is absolutely convergent for* $|x-c| < R$ *and divergent for* $|x-c| > R$*.*

R is called the radius of convergence. If $R = \infty$, then the series is absolutely convergent for all x.

The interval $(c-R, c+R)$ is referred to as the *interval of convergence*, although it's possible the interval of convergence contains one or both of the endpoints.

Proof

We will prove the theorem for $c = 0$ by showing that if the series converges for $x = x_0$, it must converge whenever $|x| < |x_0|$.

So assume the series converges for $x = x_0$. It follows that $a_n x_0^n \to 0$ as $n \to \infty$, since otherwise the series could not converge. It follows that there is some bound B such that $|a_n x_0^n| < B$ for all n.

Proof (Continued)

Now let $|x| < |x_0|$ and examine the magnitude of $a_n x^n$. $|a_n x^n| = |a_n x_0^n| \cdot \left|\frac{x}{x_0}\right|^n \leq B\left|\frac{x}{x_0}\right|^n$.

Since $\sum_{n=0}^{\infty} B\left|\frac{x}{x_0}\right|^n$ is a geometric series with common ratio $\left|\frac{x}{x_0}\right| < 1$, it must converge. By the Comparison Test, $\sum_{n=0}^{\infty} |a_n x^n| < \infty$, and thus $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent.

Finding the Radius of Convergence

For most power series, the easiest way to determine the radius of convergence is to use the ratio test.

Given a series $\sum_{n=0}^{\infty} a_n x^n$, we calculate $\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}x\right|$.

We find the limit as $n \to \infty$ and find the values of x for which the limit is less than or equal to 1.

One complication that can occur is that some coefficients a_n equal 0. In this case, we look at the ratios of the adjacent terms that actually appear.

Algebra and Calculus of Power Series

Within their intervals of convergence, although possibly not at the endpoints, power series may be added, subtracted, multiplied, differentiated and integrated in the natural way. Specifically:

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ in some open interval, in the sense that the power series converge to $A(x)$ and $B(x)$ in the interval. Then:

Algebra and Calculus of Power Series

- $A(x) + B(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$
- $A(x) - B(x) = \sum_{n=0}^{\infty} (a_n - b_n) x^n$
- $A(x)B(x) = \sum_{n=0}^{\infty} (\sum_{i=0}^{n} a_i b_{n-i}) x^n$
- $A'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$
- $\int A(x)\, dx = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} + k$

Example: $\ln(1 + x)$

Each of the following calculations can be done, based on the properties of power series, whenever $|x| < 1$:

Start with $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$.

Replace x by $-x$ to get $1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}$.

Integrate to get $(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots) + k = \ln(1 + x)$.

Plugging in $x = 0$, we find $k = 0$ to obtain $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$.

The power series also converges (by the Alternating Series Test) for $x = 1$, giving $\ln 2$ as the sum of the *Alternating Harmonic Series*:

$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$.

arctan

We can also obtain a power series converging to the arctangent function by starting with the power series $1 - x + x^2 - x^3 + \cdots = \dfrac{1}{1+x}$. All the following calculations again hold for $|x| < 1$.

Replacing x by x^2 gives $1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots = \dfrac{1}{1+x^2}$.

Integrating: $(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots) + k = \arctan x$.

Plugging in $x = 0$ yields $k = 0$, so $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$ for $|x| < 1$.

Calculation of π

$\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$ for $|x| < 1$.

Here, too, the series actually converges for $x = 1$ and converges to $\arctan 1$. Since $\arctan 1 = \dfrac{\pi}{4}$, we get the interesting series expansion:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots.$$

Obtaining a Series Which Converges to a Given Function

Suppose we have a function $f(x)$ and want to find a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ which converges to $f(x)$ in some interval. It turns out there's just one possible choice, which is called the Taylor Series centered at $x = c$. To see this, assume $f(x)$ possesses as many derivatives as we need and $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots.$

Obtaining a Power Series

$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots.$

If we try to evaluate the series for $x = 0$, we obviously get a_0, so $f(0) = a_0$ or $a_0 = f(0)$. In other words, there's just one possibility for a_0.

Differentiating, $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$, so $f'(0) = a_1$ or $a_1 = f'(0)$.

Differentiating again, $f''(x) = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \ldots$, so $f''(0) = 2a_2$ or $a_2 = \dfrac{f''(0)}{2}$.

Going one step further, $f'''(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + 5 \cdot 4 \cdot 3a_5 x^2 + 6 \cdot 5 \cdot 4a_6 a^3 + \ldots$, so $f'''(0) = 3 \cdot 2a_3$ or $a_3 = \dfrac{f'''(0)}{3 \cdot 2}$.

The Taylor Series

Summarizing the results so far:

$$\begin{aligned} a_0 &= f(0) \\ a_1 &= f'(0) \\ a_2 &= \frac{f''(0)}{2} \\ a_3 &= \frac{f'''(0)}{3 \cdot 2} \end{aligned}$$

One suspects $a_4 = \dfrac{f^{(4)}(0)}{4 \cdot 3 \cdot 2}$, which may be written as $a_4 = \dfrac{f^{(4)}(0)}{4!}$.

One thus suspects that, in general, $a_n = \dfrac{f^{(n)}(0)}{n!}$ and this is indeed the case. This leads to the following definition of a Taylor Series:

Taylor Series

Definition 28 (Taylor Series). *The Taylor Series for a function* $f(x)$*, centered at* $x = c$*, is defined as* $T(x) = T_f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

Most of the time, we will center Taylor Series at 0, in which case the formula simplifies to $T(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. These series are also known as Maclaurin Series.

Taylor Series

We have shown that if a power series converges to a function, it must be the Taylor Series. On the other hand, the Taylor Series for a function does not always have to converge to that function although it often will ... otherwise we wouldn't bother with them.

The series previously shown to converge to $\dfrac{1}{1-x}$, $\ln(1+x)$ and $\arctan x$ for $|x| < 1$ therefore must be the Taylor Series, centered at 0, for those functions.

Taylor Series for the Exponential Function

Let $f(x) = \exp(x) = e^x$. Since all the derivatives of the exponential function are the same, we have $f^{(n)}(x) = e^x$ for all n and thus $f^{(n)}(0) = e^0 = 1$ for all n.

The Taylor Series is thus $T(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

We will show this converges for all x and it converges to e^x for all x, so we may write

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots.$$

Radius of Convergence

To show where the Taylor Series for e^x converges, we use the Ratio Test.

$$\left|\frac{(n+1)^{\text{st}} \text{ term}}{n^{\text{th}} \text{ term}}\right| = \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right|$$

$$= \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right| \to 0$$

as $n \to \infty$.

By the Ratio Test, it follows that the series converges for all x.

Another Derivation of the Taylor Series

There's a second way of coming up with the formula for a Taylor Series which naturally leads to an estimate for the error if one estimates the value of the function by a partial sum of the Taylor Series.

Definition 29 (Taylor Polynomial). *The n^{th} partial sum of a Taylor Series $T(x)$ is denoted by $T_n(x)$ and is called a Taylor Polynomial.*

Consider $\int_0^x f'(t)\,dt$. Using the Fundamental Theorem of Calculus, $\int_0^x f'(t)\,dt = f(t)\Big|_0^x = f(x) - f(0)$.

It follows we can write $f(x) = f(0) + \int_0^x f'(t)\,dt$.

If we repeatedly integrate by parts, we can obtain the terms of the Taylor Series.

Integrating By Parts

$$f(x) = f(0) + \int_0^x f'(t)\,dt$$

We'll use the version $\int uv'\,dt = uv - \int u'v\,dt$, taking
$u = f'(t)$, $v' = 1$,
so $u' = f''(t)$, $v = t - x$.

Note the trick here: We could take $v = t$, but it turns out that doesn't work well, while taking $v = t - x$ works very nicely, and we obtain:

$$\int_0^x f'(t)\,dt = f'(t)(t-x)\Big|_0^x - \int_0^x f''(t)(t-x)\,dt$$

$$= f'(0)x - \int_0^x f''(t)(t-x)\,dt,$$

so

$f(x) = f(0) + f'(0)x - \int_0^x f''(t)(t-x)\,dt.$

Continuing

In order to have a sum rather than a di�erence, we'll rewrite that as $f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t)\,dt.$

Now integrate by parts again, taking $u = f''(t)$, $v' = x - t$, so $u' = f'''(t)$, $v = -\frac{(x-t)^2}{2}$, and

$$\int_0^x f''(t)(x-t)\,dt$$

$$= f''(t)\left(-\frac{(x-t)^2}{2}\right)\Big|_0^x - \int_0^x f'''(t)\left(-\frac{(x-t)^2}{2}\right)dt$$

$$= \frac{f''(0)x^2}{2} + \frac{1}{2}\int_0^x f'''(t)(x-t)^2\,dt$$

.

Continuing

We thus have

$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{1}{2}\int_0^x f'''(t)(x-t)^2\,dt.$

If we carry out another step, we obtain:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f'''(0)x^3}{3\cdot 2} + \frac{1}{3\cdot 2}\int_0^x f^{(4)}(t)(x-t)^3\,dt.$$

We can continue indefinitely, obtaining the following following result.

Taylor's Theorem

Theorem 17 (Taylor's Theorem)**.** *If $f(x)$ has sufficient derivatives in an interval to evaluate all the terms needed, then $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the n^{th} degree Taylor Polynomial for $f(x)$ centered at $x = c$ and $R_n(x) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\,dt.$*

$R_n(x)$ is called the *remainder term* and can be thought of as the error involved if one uses $T_n(x)$, the n^{th} degree Taylor Polynomial, to estimate $f(x)$. It gives us a way of determining whether the Taylor Series for a function converges to that function, since

Convergence

Corollary 18. *The Taylor Series converges to* $f(x)$ *if and only if* $\lim_{n\to\infty} R_n(x) = 0$*.*

Suppose we can find a bound B on $|f^{(n+1)}(t)|$ on the interval $[c, x]$. *We are tacitly assuming* $c < x$*, but the result we get will still hold if* $x < c$*.* It follows that

$$\left|\int_c^x f^{(n+1)}(t)(x-t)^n\,dt\right| \le B\int_c^x (x-t)^n\,dt = B\left[-\frac{(x-t)^{n+1}}{n+1}\right]_c^x = B\frac{(x-c)^{n+1}}{n+1}.$$

It follows that $|R_n(x)| \le \dfrac{1}{n!}\cdot B\dfrac{(x-c)^{n+1}}{n+1} = \dfrac{B(x-c)^{n+1}}{(n+1)!}$.

The Remainder Term

We have shown:

Theorem 19. *If* $|f^{(n+1)}|$ *is bounded by* B *on an interval containing* c *and* x*, then* $|R_n(x)| \le \dfrac{B|x-c|^{n+1}}{(n+1)!}$*.*

Corollary 20. *If there is a uniform bound on all the derivatives of a function on an interval, then the Taylor Series for that function must converge to that function at all points on that interval.*

One immediate consequence is that the Taylor Series for e^x, $\sin x$ and $\cos x$ all converge to those functions everywhere!

Parametric Equations

We sometimes have several equations sharing an independent variable. In those cases, we call the independent variable a *parameter* and call the equations *parametric equations*. In many cases, the domain of the parameter is restricted to an interval.

Example: Motion of a Projectile

Suppose a projectile is launched at an initial speed v_0, from a height h_0, at an angle θ with the horizontal. It's natural to consider the horizontal distance and the height of the projectile separately. Let

t represent time,
x represent the horizontal distance from the launching spot,
y represent the height, and
g the acceleration due to gravity, in the appropriate units.

In the English system, $g \approx -32.2$ and in the metric system $g \approx -9.8$. In each case, g is negative since gravity acts in the downward, or negative, direction.

Analyzing Horizontal Motion

If one was looking at the projectile from above and had no depth perception, it would look as if the projectile was travelling in a straight line at a constant speed equal to $v_0 \cos\theta$.

Since the speed is constant, it should be clear that

$x = v_0 \cos\theta t$.

Analyzing Vertical Motion

If one looked at the projectile from behind, in the plane of its motion, and had no depth perception, it would look as if the projectile was first going straight up and then falling, with an initial upward speed of $v_0 \sin\theta$ but subject to gravity causing an acceleration g.

If we let v_y represent the speed at which the projectile appears to be rising, $\frac{dv_y}{dt} = g$, so $v_y = \int g\,dt = gt + c$ for some constant $c \in \mathbb{R}$.

Analyzing Vertical Motion

Since $v_y = v_0 \sin\theta$ when $t = 0$, we have $v_0 \sin\theta = g \cdot 0 + c$, so $c = v_0 \sin\theta$ and $v_y = gt + v_0 \sin\theta$.

Since $v_y = \frac{dy}{dt}$, it follows that $y = \int gt + v_0 \sin\theta\,dt$, so $y = \frac{1}{2}gt^2 + v_0 \sin\theta t + k$ for some $k \in \mathbb{R}$.

Since $y = y_0$ when $t = 0$, it follows that

$y_0 = \frac{1}{2}g \cdot 0^2 + v_0 \sin\theta \cdot 0 + k$,

so $k = y_0$ and $y = \frac{1}{2}gt^2 + v_0 \sin\theta t + y_0$.

Putting It Together

We thus have the parametric equations:

$x = v_0 \cos t$

$y = \frac{1}{2}gt^2 + v_0 \sin\theta t + y_0$

These equations will hold until the projectile strikes something.

The Unit Circle

The unit circle is another natural example of the use of parametric equations, since the two coordinates of a point on the circle both depend on the angle with the horizontal made by the radius through the point.

Indeed, by definition, if we let (x, y) be the coordinates of the point on the unit circle for which the angle referred to above is θ, then $x = \cos\theta$ and $y = \sin\theta$. Thus,

$x = \cos\theta$
$y = \sin\theta$
$0 \leq \theta \leq 2\pi$
is a pair of parametric equations describing the circle. Indeed, as θ goes from 0 to 2π, the point (x, y) traverses the circumference of the circle.

Slopes of Tangents

Assume we have parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ and both $f(t)$ and $g(t)$ are di�erentiable. If we're at a point where $f'(t) \neq 0$, then there is some interval containing that point in which $f(t)$ is monotonic and will have a local inverse. In that interval, we may write $t = f^{-1}(x)$.

We may use the Chain Rule to obtain $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$, and thus $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

This enables us to find the slope of the tangent to the graph of the parametric equations at any point where $f'(t) \neq 0$.

The points where $f'(t) = 0$ are points where the tangent lines are vertical, so that's not a tremendous problem.

Arc Length

Given parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, $\{(x, y) | x = f(t), y = g(t), a \leq t \leq b\}$ will generally form a curve. If $f(t)$ and $g(t)$ are di�erentiable, we can find its length.

Let
n be a positive integer,
$\Delta t = \dfrac{b-a}{n}$,
$t_k = a + k\Delta t$,
$x_k = f(t_k)$,
$y_k = g(t_k)$,
s = the length of the curve,
Δs_k = the length of the portion of the curve for $t_{k-1} \leq t \leq t_k$.

Clearly, $s = \sum_{k=1}^{n} \Delta s_k = \Delta s_1 + \Delta s_2 + \Delta s_3 + \cdots + \Delta s_n$.

Arc Length

We can approximate Δs_k by the length of the line segment connecting (x_{k-1}, y_{k-1}) and (x_k, y_k). Using the distance formula, we approximate $\Delta s_k \approx \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$.

This is precisely what was done in approximating arc length when a curve was the graph of an ordinary function. What will differ for parametric curves will be the way we estimate $\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$.

Using the Mean Value Theorem, there is some $\xi_k \in [t_{k-1}, t_k]$ such that $x_k - x_{k-1} = f'(\xi_k)\Delta t$.

Similarly, there is some $\eta_k \in [t_{k-1}, t_k]$ such that $y_k - y_{k-1} = g'(\eta_k)\Delta t$.

Thus,
$\Delta s_k \approx \sqrt{(f'(\xi_k)\Delta t_k)^2 + (g'(\eta_k)\Delta t)^2}$

$$s = \sum_{k=1}^{n} \Delta s_k$$

$\Delta s_k \approx \sqrt{(f'(\xi_k)\Delta t_k)^2 + (g'(\eta_k)\Delta t)^2} = \sqrt{(f'(\xi_k)^2 + g'(\eta_k)^2)(\Delta t)^2} = \sqrt{f'(\xi_k)^2 + g'(\eta_k)^2}\Delta t$.

There won't be much di□erence between $g'(\eta_k)$ and $g'(\xi_k)$ if Δt is small. Since we're only approximately the arc length anyway, we may write $\Delta s_k \approx \sqrt{f'(\xi_k)^2 + g'(\xi_k)^2}\Delta t$

We thus can approximate
$s \approx \sum_{k=1}^{n} \sqrt{f'(\xi_k)^2 + g'(\xi_k)^2}\Delta t$.

$$s \approx \sum_{k=1}^{n} \sqrt{f'(\xi_k)^2 + g'(\xi_k)^2}\Delta t$$

The sum is a Riemann Sum for the function $\sqrt{f'(t)^2 + g'(t)^2}$, so we may expect
$s = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt$.

This may also be written in the form
$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

For curves described by ordinary equations, this formula for arc length reduces to the familiar one. Suppose we have a curve $y = f(x)$, $a \le x \le b$.

Every such function has a *Canonical Parametrization*:

$x = t$
$y = f(t)$
$a \leq t \leq b$

Since $\dfrac{dx}{dt} = \dfrac{d}{dt}(t) = 1$, while $\dfrac{dy}{dt} = \dfrac{d}{dt}(f(t)) = f'(t)$, we may write

$s = \int_a^b \sqrt{1^2 + f'(t)^2}\,dt = \int_a^b \sqrt{1 + f'(t)^2}\,dt.$

This is the formula previously derived for curves given by ordinary functions.

Circumference of a Circle

The arc length formula can be used to derive the formula for the circumference of a circle.

A circle of radius r, centered at the origin, may be parametrized by

$x = r\cos t$
$y = r\sin t$
$0 \leq t \leq 2\pi.$

We have $\dfrac{dx}{dt} = -r\sin t$, $\dfrac{dy}{dt} = r\cos t$, so $\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} = \sqrt{(-r\sin t)^2 + (r\cos t)^2} = \sqrt{r^2\sin^2 t + r^2\cos^2 t} = \sqrt{r^2(\sin^2 t + \cos^2 t)} = \sqrt{r^2 \cdot 1} = r.$

Circumference of a Circle

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = r$$

$s = \int_0^{2\pi} r\,dt = rt\Big|_0^{2\pi} = r \cdot 2\pi - r \cdot 0 = 2\pi r.$

This calculation is really circular, since π is defined as the ratio of the circumference of a circle to its diameter.

The Dot Product

$$< a, b > \cdot < c, d > = ac + bd$$

$$< a, b, c > \cdot < d, e, f > = ad + be + cf$$

The Dot Product and Angle Between Vectors

Look at a triangle formed by vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{v} - u$ going from the tip of $\mathbf{u}$ to the tip of $\mathbf{v}$.

Write $\mathbf{u} =< a, b >$, $\mathbf{v} =< c, d >$, so $\mathbf{v} - u =< c - a, d - b >$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

Apply the *Law of Cosines*:

$|\mathbf{v} - \mathbf{u}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}| \cdot |\mathbf{v}| \cos\theta$.

We get $(c-a)^2 + (d-b)^2 = (a^2+b^2) + (c^2+d^2) - 2|\mathbf{u}| \cdot |\mathbf{v}| \cos\theta$.

Simplifying, $(c^2 - 2ac + a^2) + (d^2 - 2bd + b^2) = a^2 + b^2 + c^2 + d^2 - 2|\mathbf{u}| \cdot |\mathbf{v}| \cos\theta$, so

$-2ac - 2bd = -2|\mathbf{u}| \cdot |\mathbf{v}| \cos\theta$.

Dividing both sides by -2, we get $ac + bd = |\mathbf{u}| \cdot |\mathbf{v}| \cos\theta$. From this, we see that the connection between the dot product and the angle between the vectors:

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cos\theta$.

Extra Credit: Show that this is also true in three dimensions.

Orthogonality

Orthogonal is a way of saying perpendicular.

The dot product gives an easy way of determining whether two vectors are *orthogonal*–just calculate the dot product of the vectors and check whether it's equal to 0.

Properties of the Dot Product

- Closure: No. The dot product of two vectors is a scalar, not a vector.
- Commutative Law: Yes. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
- Associative Law: No.
- Existence of an identity: No.
- Existence of an inverse: No.
- Distributive Law: Yes.
  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

The Standard Basis Vectors

The unit (length 1) vectors in the directions of the coordinate axis are called the *standard basis vectors* and denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

In two dimensions: $\mathbf{i} = <1, 0>$, $\mathbf{j} = <0, 1>$.

In three dimensions: $\mathbf{i} = <1, 0, 0>$, $\mathbf{j} = <0, 1, 0>$, $\mathbf{k} = <0, 0, 1>$.

Any vector can be written in terms of the standard basis vectors:

$<a, b> = a\mathbf{i} + b\mathbf{j}$, $<a, b, c> = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

The Dot Product: $(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = ad + be + cf$.

The Cross Product

The Cross Product $\mathbf{u} \times \mathbf{v}$ is designed so that (a) the product of two unit vectors is a unit vector orthogonal to the two multiplicands and (b) the three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ form a *right hand triple*. It's also designed to satisfy the distributive law $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.

Cross Products of Standard Basis Vectors

- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$
- $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
- $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$
- $\mathbf{i} \times \mathbf{i} = \mathbf{0}$, $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, $\mathbf{k} \times \mathbf{k} = \mathbf{0}$

Formula for the Cross Product

To derive a formula for the cross product in general, repeatedly apply the distributive law to the product.

$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) =$

$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times d\mathbf{i} + (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times e\mathbf{j} + (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times f\mathbf{k} =$

$a\mathbf{i} \times d\mathbf{i} + a\mathbf{i} \times e\mathbf{j} + a\mathbf{i} \times f\mathbf{k} + b\mathbf{j} \times d\mathbf{i} + b\mathbf{j} \times e\mathbf{j} + b\mathbf{j} \times f\mathbf{k} + c\mathbf{k} \times d\mathbf{i} + c\mathbf{k} \times e\mathbf{j} + c\mathbf{k} \times f\mathbf{k} =$

$ad\mathbf{i} \times \mathbf{i} + ae\mathbf{i} \times \mathbf{j} + af\mathbf{i} \times \mathbf{k} + bd\mathbf{j} \times \mathbf{i} + be\mathbf{j} \times \mathbf{j} + bf\mathbf{j} \times \mathbf{k} + cd\mathbf{k} \times \mathbf{i} + ce\mathbf{k} \times \mathbf{j} + cf\mathbf{k} \times \mathbf{k} =$

$ad\mathbf{0} + ae\mathbf{k} - af\mathbf{j} - bd\mathbf{k} + be\mathbf{0} + bf\mathbf{i} + cd\mathbf{j} - ce\mathbf{i} + cf\mathbf{0} =$

$(bf - ce)\mathbf{i} + (cd - af)\mathbf{j} + (ae - bd)\mathbf{k}$

Definition of the Cross Product

Definition 30 (Cross Product). $(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = (bf - ce)\mathbf{i} + (cd - af)\mathbf{j} + (ae - bd)\mathbf{k}$.

Symbolically:

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix}$$

Properties of the Cross Product

- Closure. Yes. The cross product of vectors is a vector.
- Commutative Law. No. The cross product is *anti-commutative*: $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$.
- Associative Law. No. But $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$. This is called the *Triple Product* and its absolute value is equal to the volume of the parallelopiped determined by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.
- Existence of an Identity. No.
- Existence of an Inverse. No.
- Distributive Law. Yes.

  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

  $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

Other Properties of the Cross Product

- $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
- $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$.
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$.