# Parametric Equations

We sometimes have several equations sharing an independent variable. In those cases, we call the independent variable a *parameter* and call the equations *parametric equations*. In many cases, the domain of the parameter is restricted to an interval.

## Example: Motion of a Projectile

Suppose a projectile is launched at an initial speed $v_0$, from a height $h_0$, at an angle $\theta$ with the horizontal. It's natural to consider the horizontal distance and the height of the projectile separately. Let

$t$ represent time,
$x$ represent the horizontal distance from the launching spot,
$y$ represent the height, and
$g$ the acceleration due to gravity, in the appropriate units.

In the English system, $g \approx -32.2$ and in the metric system $g \approx -9.8$. In each case, $g$ is negative since gravity acts in the downward, or negative, direction.

## Analyzing Horizontal Motion

If one was looking at the projectile from above and had no depth perception, it would look as if the projectile was travelling in a straight line at a constant speed equal to $v_0 \cos\theta$.

Since the speed is constant, it should be clear that

$x = v_0 \cos\theta t$.

## Analyzing Vertical Motion

If one looked at the projectile from behind, in the plane of its motion, and had no depth perception, it would look as if the projectile was first going straight up and then falling, with an initial upward speed of $v_0 \sin\theta$ but subject to gravity causing an acceleration $g$.

If we let $v_y$ represent the speed at which the projectile appears to be rising, $\dfrac{dv_y}{dt} = g$, so $v_y = \int g\,dt = gt + c$ for some constant $c \in \mathbb{R}$.

Since $v_y = v_0 \sin\theta$ when $t = 0$, we have $v_0 \sin\theta = g \cdot 0 + c$, so $c = v_0 \sin\theta$ and $v_y = gt + v_0 \sin\theta$.

Since $v_y = \dfrac{dy}{dt}$, it follows that $y = \int gt + v_0 \sin\theta\,dt$, so $y = \dfrac{1}{2}gt^2 + v_0 \sin\theta t + k$ for some $k \in \mathbb{R}$.

Since $y = y_0$ when $t = 0$, it follows that
$y_0 = \frac{1}{2} g \cdot 0^2 + v_0 \sin\theta \cdot 0 + k$,
so $k = y_0$ and $y = \frac{1}{2} g t^2 + v_0 \sin\theta t + y_0$.

## Putting It Together

We thus have the parametric equations:
$x = v_0 \cos t$
$y = \frac{1}{2} g t^2 + v_0 \sin\theta t + y_0$

These equations will hold until the projectile strikes something.

## The Unit Circle

The unit circle is another natural example of the use of parametric equations, since the two coordinates of a point on the circle both depend on the angle with the horizontal made by the radius through the point.

Indeed, by definition, if we let $(x, y)$ be the coordinates of the point on the unit circle for which the angle referred to above is $\theta$, then $x = \cos\theta$ and $y = \sin\theta$. Thus,

$x = \cos\theta$
$y = \sin\theta$
$0 \leq \theta \leq 2\pi$
is a pair of parametric equations describing the circle. Indeed, as $\theta$ goes from 0 to $2\pi$, the point $(x, y)$ traverses the circumference of the circle.

## Slopes of Tangents

Assume we have parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ and both $f(t)$ and $g(t)$ are differentiable. If we're at a point where $f'(t) \neq 0$, then there is some interval containing that point in which $f(t)$ is monotonic and will have a local inverse. In that interval, we may write $t = f^{-1}(x)$.

We may use the Chain Rule to obtain $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, and thus $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

This enables us to find the slope of the tangent to the graph of the parametric equations at any point where $f'(t) \neq 0$.

The points where $f'(t) = 0$ are points where the tangent lines are vertical, so that's not a tremendous problem.

## Arc Length

Given parametric equations $x = f(t)$, $y = g(t)$, $a \le t \le b$, $\{(x,y)|x = f(t), y = g(t), a \le t \le b\}$ will generally form a curve. If $f(t)$ and $g(t)$ are differentiable, we can find its length.

Let
$n$ be a positive integer,
$\Delta t = \dfrac{b-a}{n}$,
$t_k = a + k\Delta t$,
$x_k = f(t_k)$,
$y_k = g(t_k)$,
$s =$ the length of the curve,
$\Delta s_k =$ the length of the portion of the curve for $t_{k-1} \le t \le t_k$.

Clearly, $s = \sum_{k=1}^{n} \Delta s_k = \Delta s_1 + \Delta s_2 + \Delta s_3 + \cdots + \Delta s_n$.

$$s = \sum_{k=1}^{n} \Delta s_k$$

We can approximate $\Delta s_k$ by the length of the line segment connecting $(x_{k-1}, y_{k-1})$ and $(x_k, y_k)$. Using the distance formula, we approximate $\Delta s_k \approx \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$.

*This is precisely what was done in approximating arc length when a curve was the graph of an ordinary function. What will differ for parametric curves will be the way we estimate* $\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$.

Using the Mean Value Theorem, there is some $\xi_k \in [t_{k-1}, t_k]$ such that $x_k - x_{k-1} = f'(\xi_k)\Delta t$.

Similarly, there is some $\eta_k \in [t_{k-1}, t_k]$ such that $y_k - y_{k-1} = g'(\eta_k)\Delta t$.

Thus,
$\Delta s_k \approx \sqrt{(f'(\xi_k)\Delta t_k)^2 + (g'(\eta_k)\Delta t)^2} = \sqrt{(f'(\xi_k)^2 + g'(\eta_k)^2)(\Delta t)^2} = \sqrt{f'(\xi_k)^2 + g'(\eta_k)^2}\Delta t$.

There won't be much difference between $g'(\eta_k)$ and $g'(\xi_k)$ if $\Delta t$ is small. Since we're only approximately the arc length anyway, we may write
$\Delta s_k \approx \sqrt{f'(\xi_k)^2 + g'(\xi_k)^2}\Delta t$

We thus can approximate
$$s \approx \sum_{k=1}^{n} \sqrt{f'(\xi_k)^2 + g'(\xi_k)^2}\Delta t.$$

The sum is a Riemann Sum for the function $\sqrt{f'(t)^2 + g'(t)^2}$, so we may expect
$$s = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

This may also be written in the form
$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

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For curves described by ordinary equations, this formula for arc length reduces to the familiar one. Suppose we have a curve $y = f(x)$, $a \le x \le b$.

Every such function has a *Canonical Parametrization*:

$x = t$
$y = f(t)$
$a \le t \le b$

Since $\frac{dx}{dt} = \frac{d}{dt}(t) = 1$, while $\frac{dy}{dt} = \frac{d}{dt}(f(t)) = f'(t)$, we may write
$s = \int_a^b \sqrt{1^2 + f'(t)^2}\, dt = \int_a^b \sqrt{1 + f'(t)^2}\, dt.$

This is the formula previously derived for curves given by ordinary functions.

## Circumference of a Circle

The arc length formula can be used to derive the formula for the circumference of a circle.

A circle of radius $r$, centered at the origin, may be parametrized by

$x = r\cos t$
$y = r\sin t$
$0 \le t \le 2\pi.$

We have $\frac{dx}{dt} = -r\sin t$, $\frac{dy}{dt} = r\cos t$, so $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-r\sin t)^2 + (r\cos t)^2} = \sqrt{r^2\sin^2 t + r^2\cos^2 t} = \sqrt{r^2(\sin^2 t + \cos^2 t)} = \sqrt{r^2 \cdot 1} = r$, so

$s = \int_0^{2\pi} r\, dt = rt\Big|_0^{2\pi} = r \cdot 2\pi - r \cdot 0 = 2\pi r.$

*This calculation is really circular, since $\pi$ is defined as the ratio of the circumference of a circle to its diameter.*