Mathematics 116 SOLUTIONS
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1. Use Simpson's Rule with $n = 100$ to estimate $\int_0^1 \frac{2t}{t^2+1}\,dt$. *You may use a spreadsheet or a procedural computer language (generally, a language whose name doesn't include the word visual) to do the calculations. The spreadsheet or source for the program should be annotated enough so that the calculations are clear.*

   Solution: The following is a quick-and-dirty C program that will do the calculation.

```
#include <stdio.h>
float f (float x)
{ return 2.0*x/(x*x+1) ; }
main()
{
  int a=0,b=1,n=100,i,factor=4;
  float sum,x,h;
  h=(float)(b-a)/n;
  x=a+h;
  sum=f(a)+f(b);
  for(i=1;i<n; i++)
    {
      sum += factor*f(x);
      x+=h;
      factor=6-factor;
    }
  sum*=h/3;
  printf("\nThe integral is approximately %1.16f\n",sum);
}
```

   The output from this program is

```
The integral is approximately 0.6931470036506653
```

2. Use the formula given in class for the maximum error using Simpson's Rule to obtain a bound on the error in your calculations.

   Solution: Letting $f(t) = \dfrac{2t}{t^2+1}$, we get:

   $f\ (t) = \dfrac{2(1-t^2)}{(t^2+1)^2}$

   $f\ \ (t) = \dfrac{4t(t^2-3)}{(t^2+1)^3}$

   $f\ \ \ (t) = -\dfrac{12(t^4-6t^1+1)}{(t^2+1)^4}$

   $f^{(4)}(t) = \dfrac{48t(t^4-10t^2+5)}{(t^2+1)^5}.$

   On the interval from 0 to 1, $t^1 + 1 \geq 1$, and $|t^4 - 10t^2 + 5| \leq \max(t^4 + 5, 10t) = 10$, so clearly $|f^{(4)}(t)| \leq \dfrac{48 \cdot 1 \cdot 10}{1^5} = 480$.

   Using the bound $|E_S| \leq \dfrac{K(b-a)^5}{180n^4}$, where K is a bound on the fourth derivative of the integrand, we get $|E_S| \leq \dfrac{480(1-0)^5}{180 \cdot 100^4} = \dfrac{8 \cdot 10^{-8}}{3} \approx 2.67 \cdot 10^{-8}$

3. Evaluate $\displaystyle\int_0^1 \frac{2t}{t^2+1}\,dt$ exactly.

   Solution: $\displaystyle\int_0^1 \frac{2t}{t^2+1}\,dt = \ln(1+t^2)\Big|_0^1 = \ln 2 - \ln 1 = \ln 2.$

4. Use a calculator to get a decimal approximation for the difference between the approximation you obtained using Simpson's Rule and the exact value you obtained.

   Solution: A calculator give $\ln 2 \approx 0.69314718056$.

   The difference between that and the approximation obtained using Simpson's Rule is $|0.6931470036506653 - 0.69314718056| \approx 1.76909334737 \times 10^{-7} = 0.000000176909334737$.

5. Compare that difference, effectively your error using Simpson's Rule, to the theoretical maximum error you determined.

   Solution: This is considerably larger than the theoretical maximum, demonstrating the effect of roundoff error.

   Rerunning the program using double precision led to an approximation of 0.6931471813934376, with an error of $8.33437541203 \cdot 10^{-10}$, which is only about three-tenths of a percent of the theoretical maximum.