Mathematics 115 — SOLUTIONS

Professor Alan H. Stein

Due Friday, April 20, 2007 — This problem set is worth 50 points.

1. (a) Define relative minimum.
   (b) Define strictly increasing.
   (c) Using plain language and avoiding all use of mathematical notation, write down a strategy for sketching graphs.
   (d) Using plain language and avoiding all use of mathematical notation, write down a strategy for tackling optimization problems.

   Solution:

   (a) A function f has a relative minimum at c if there is some open interval containing c such that $f(x) \geq f(c)$ for all x in the interval.

   (b) A function f is strictly increasing on an interval if $f(a) < f(b)$ for every pair a, b of points in the interval with $a < b$.

   (c) Calculate the first and second derivatives of the function, factor each completely, analyze their signs and use that analysis to analyze monotonicity and concavity for the function. Find all critical points (where the derivative is either 0 or undefined) as well as all points where the second derivative is either 0 or undefined. Find all values of the function at those points (where they exist) and use the information about monotonicity and concavity to sketch each portion of the graph between (and outside) each pair of those points which are adjacent. If necessary, check for horizontal and vertical asymptotes and, if it's not clear whether the graph crosses the y-axis above or below the origin, find the y-intercept.

   (d) Read the question carefully. Find all variable and unknown quantities and represent them by letters. Translate each piece of information, explicit or implied, into mathematical statements, most often equations or formulas. Solve equations. If you have the quantity to be maximized written as a function of one or more other variables, differentiate it and see when the derivative is either 0 or undefined. The extremum you are looking for must be at one of those points.

2. A function f is continuous and differentiable everywhere except where the information given implies otherwise. It is strictly increasing on $(-\infty, -3) \cup (-3, 4) \cup (8, \infty)$ and strictly decreasing on $(4, 8)$. It is concave up on $(-\infty, -3) \cup (6, \infty)$ and concave down on $(-3, 6)$. $\lim_{x \to -\infty} f(x) = 5$, $\lim_{x \to -3^-} f(x) = \infty$, $\lim_{x \to -3^+} f(x) = -\infty$, $f(0) = -3$, $f(4) = 7$, $f(6) = 5$, $f(8) = 2$. Sketch its graph and completely identify all relative and absolute extrema, all points of inflection, all discontinuities and all asymptotes.

   Solution: There is a relative maximum at 4 and a relative minimum at 8. There are no absolute extrema. There is a point of inflection at 6. There is a discontinuity at -3, a vertical asymptote at -3 and a horizontal asymptote (on the left) of $y = 5$.

3. Let $f(x) = x^3 - 12x$. Sketch its graph and completely identify all relative and absolute extrema, all points of inflection, all discontinuities and all asymptotes.

   Solution:

   $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2)$. Clearly $f'(x) = 0$ if $x = \pm 2$ and f' is positive and f is increasing on $(-\infty, -2) \cup (2, \infty)$, while f' is negative and f is decreasing on $(-2, 2)$.

   $f''(x) = 6x$. Clearly $f''(x) < 0$ and f is concave down for $x < 0$ while $f''(x) > 0$ and f is concave up for $x > 0$.

   We would plot $(-2, 16)$, $(0, 0)$ and $(2, -16)$.

   Clearly, f has a relative maximum at -2 and a relative minimum at 2 but has no absolute extrema. f has a point of inflection at 0. f is continuous everywhere and has no asymptotes.

4. Let $f(x) = \sin x + \cos x$. Sketch its graph and completely identify all relative and absolute extrema, all points of inflection, all discontinuities and all asymptotes. *Extra Credit: Show how the graph can be sketched without using any Calculus.*

   Solution: It suffices to analyze f on $[0, 2\pi]$, since f is clearly periodic with period 2π.

   $f'(x) = \cos x - \sin x$. $f'(x) = 0$ when $\sin x = \cos x$, which occurs at $\pi/4$ and $5\pi/4$. Between 0 and $\pi/4$, $\cos x > \sin x$, so $f'(x) > 0$ and f is increasing. Between $\pi/4$ and $5\pi/4$, $\cos x < \sin x$, so $f'(x) < 0$ and f is decreasing. Between $5\pi/4$ and 2π, $\cos x > \sin x$ again, so $f'(x) > 0$ and f is increasing. $f(0) = \cos 0 + \sin 0 = 1$. Similarly, $f(2\pi) = 1$. $f(\pi/4) = \cos(\pi/4) + \sin(\pi/4) = 1/\sqrt{2} + 1$
   $sqrt2 = 2/\sqrt{2} = \sqrt{2}$, while $f(5/pi/4) = \cos(5\pi/4) + \sin(5\pi/4) = -1/\sqrt{2} + (-1\sqrt{2}) = -2/\sqrt{2} = -\sqrt{2}$

   $f''(x) = -\sin x - \cos x$. $f''(x) = 0$ when $\sin x = -\cos x$, which occurs at $3\pi/4$ and $7\pi/4$. Between 0 and $\pi/4$, both $\sin x$ and $\cos x$ are positive, so $f''(x)$ must be negative. Between $\pi/4$ and $3\pi/4$, $\sin x > |\cos x|$, so $f''(x)$ is negative, and thus $f''(x)$ is negative and the graph is concave down on $(0, 3\pi/4)$. Between $3\pi/4$ and $5\pi/4$, $|cosx| > |\sin x|$ and $\cos x < 0$, so $f''(x) > 0$. Between $5\pi/4$ and $7\pi/4$, $|sinx| > |\cos x|$ and $\sin x < 0$, so $f''(x) > 0$. Thus $f''(x)$ is positive and the graph is concave up between $3\pi/4$ and $7\pi/4$. Between $7\pi/4$ and 2π, $\cos x > |\sin x|$, so $f''(x) < 0$ and the graph is concave down. $f(3\pi/4) = \cos(3\pi/4) + \sin(3\pi/4) = -1/\sqrt{2} + 1/\sqrt{2} = 0$, while $f(7\pi/4) = \cos(7\pi/4) + \sin(7\pi/4) = 1/\sqrt{2} + (-1/\sqrt{2}) = 0$.

   Clearly f has an absolute and relative maximum at $\pi/4$ and an absolute and relative minimum at $5\pi/4$, while it has points of inflection at $3\pi/4$ and $7\pi/4$. These repeat every 2π. There are no discontinuities or asymptotes.

   Extra Credit: $f(x) = \cos x + \sin x = \sqrt{2}[(\sin x)(1/\sqrt{2}) + (\cos x)(1/\sqrt{2})] =$
   $\sqrt{2}[\sin x \cos \pi/4 + \cos x \sin \pi/4] = \sqrt{2}(\sin(x + \pi/4))$. *Thus the graph of f will be just like the graph of sin, just shifted left by $\pi/4$ and stretched vertically by a factor of $\sqrt{2}$.*

5. An automobile is traveling at a speed of 80 feet per second when the driver sees a young child in the road 195 feet in front. It takes a half second before he steps on the brakes as fast as he can, at which point the car slows down 20 feet per second each second until it stops. Does it hit the car? *Use only the tools of Calculus; do not use any formulas you may have memorized from other courses such as physics.*

   Solution: Let s represent the distance the car travels from the time the driver sees the young child. Let t represent the time that has elapsed since then. Let v represent the speed of the car and let a represent its acceleration. Once the brakes are applied, we know $a = -20$ and $\dfrac{dv}{dt} = a$, so $v = \int a\,dt = \int(-20)\,dt = -20t + k$ for some constant k. Since the car only starts slowing down from its original speed of 80 feet per second when $t = \frac{1}{2}$, we know $v = 80$ when $t = \frac{1}{2}$, so $80 = -20(\frac{1}{2}) + k$, $80 = -10 + k$, $k = 90$ and thus $v = -20t + 90$ when $t \geq \frac{1}{2}$.

   Similarly, once the brakes are applied, $s = \int v\,dt = \int(-20t + 90)\,dt = -10t^2 + 90t + c$ for some constant c. In the half second before the brakes are applied, the car will travel 40 feet, so $s = 40$ when $t = \frac{1}{2}$. Thus $40 = -10(\frac{1}{2})^2 + 90\frac{1}{2} + c$, $40 = -\frac{5}{2} + 45 + c$, $c = -\frac{5}{2}$ and $s = -10t^2 + 90t - \frac{5}{2}$ when $t \geq \frac{1}{2}$.

   The car will stop when $v = 0$. Solving $-20t + 90 = 0$, we get $20t = 90$, $t = \dfrac{90}{20} = \dfrac{9}{2}$. So it takes four and a half seconds before the car stops.

   When $t = \dfrac{9}{2}$, $s = -10(\frac{9}{2})^2 + 90(\frac{9}{2}) - \frac{5}{2} = 200$.

   Thus the car goes 200 feet before it stops. Since the youth is only 195 feet down the road, the car hits the child.

6. Find the point on the hyperbola $xy = 1$ which is closest to the origin. *The conclusion may seem obvious, but it must be justified.*

   Solution: By the distance formula, the distance between a point (x, y) on the graph and the origin is $\sqrt{x^2 + y^2}$. Since a distance is minimal precisely when its square is minimal, it suffices to minimize $z = x^2 + y^2$.

   The equation $xy = 1$ of the hyperbola defines y implicitly as a function of x. Differentiating implicitly, we get $\frac{d}{dx}(xy) = \frac{d}{dx}(1)$, $x\frac{dy}{dx} + y \cdot 1 = 0$, $x\frac{dy}{dx} + y = 0$.

   From $z = x^2 + y^2$, we get $\frac{dz}{dx} = \frac{d}{dx}(x^2 + y^2) = 2x + 2x\frac{dy}{dx}$. Since any extremum must occur where the derivative is 0 or undefined, we look for points where $2x + 2x\frac{dy}{dx} = 0$, or $x + x\frac{dy}{dx} = 0$.

   From $x\frac{dy}{dx} + y = 0$ and $x + x\frac{dy}{dx} = 0$, we immediately see $x = y$. Plugging $y = x$ into the equation $xy = 1$, we get $x^2 = 1$, $x = \pm 1$.

   Thus the only possible extrema occur at $(-1, -1)$ and $(1, 1)$. Those two points are obviously tied for being closest to the origin, since traveling from either one in either possible direction clearly brings one arbitrarily far from the origin.

7. An open rectangular box has a square base and a capacity of 250 cubic inches. The material used for the bottom costs one cent per square inch, while the material used for the sides only costs half a cent per square inch. The box was designed so the cost for the material would be as little as possible. How much did the material cost?

   Solution: Let x be the length of the square base and let y be the height. Since the volume is 250 square inches, we know $x^2y = 250$.

   Let C be the cost.

   The bottom has an area x^2. Since the bottom costs one cent per square inch, the cost of the bottom is also numerically equal to x^2.

   Each side has an area xy. Since each side costs a half cent per square inch, the cost of each side is numerically equal to $\frac{1}{2}xy$. Since there are four sides, the total cost for the sides is $4(\frac{1}{2}xy) = 2xy$.

   We thus have $C = x^2 + 2xy$.

   We could solve $x^2y = 250$ for y and get C explicitly in terms of x alone, but it's easier to use implicit differentiation, treating y as a function of x.

   Differentiating implicitly, we get $\frac{d}{dx}(x^2y) = \frac{d}{dx}(250)$, $x^2\frac{dy}{dx}+2xy = 0$, $x(x\frac{dy}{dx}+2y) = 0$. Since $x = 0$ is impossible, it follows that $x\frac{dy}{dx} + 2y = 0$.

   We also calculate $\frac{d}{dx}(C) = \frac{d}{dx}(x^2 + 2xy) = 2x + 2x\frac{dy}{dx} + 2y$. Since, at an extrema, we must have $\frac{dC}{dx} = 0$, we get $2x + 2x\frac{dy}{dx} + 2y = 0$, or $x + x\frac{dy}{dx} + y = 0$.

   We therefore must have $x^2y = 250$, $x\frac{dy}{dx} + 2y = 0$, $x + x\frac{dy}{dx} + y = 0$.

   The latter two equations imply $x = y$, so we have $x^2 \cdot x = 250$, $x^3 = 250$, $x = \sqrt[3]{250} = 5\sqrt[3]{2}$. We thus also have $y = 5\sqrt[3]{2}$ and the cost is $C = x^2+2xy = (5\sqrt[3]{2})^2+2(5\sqrt[3]{2})(5\sqrt[3]{2}) = 3(5\sqrt[3]{2})^2 = 75\sqrt[3]{4} \approx 119.055078898$ cents, or approximately \$1.19.

8. Calculate $\int \sin^4\theta \cos\theta \, d\theta$.

   Solution: Using Substitution, let $u = \sin\theta$, $\dfrac{du}{d\theta} = \cos\theta$, $d\theta = \dfrac{du}{\cos\theta}$, so $\int \sin^4\theta\cos\theta\, d\theta =$ $\int u^4 \cos\theta \cdot \dfrac{du}{\cos\theta} = \int u^4\, du = \dfrac{u^5}{5} = \dfrac{\sin^5\theta}{5} + k$.

9. Calculate $\displaystyle\int_{\pi/6}^{\pi/3} \sin x\, dx$.

   Solution: $\displaystyle\int_{\pi/6}^{\pi/3} \sin x\, dx = -\cos x\Big|_{\pi/6}^{\pi/3} = -\cos(\pi/3) - (-\cos\pi/6) = \cos(\pi/6) - \cos(\pi/3)$ $= \dfrac{\sqrt{3}}{2} - \dfrac{1}{2} = \dfrac{\sqrt{3}-1}{2}$.

10. Let $f(x) = \displaystyle\int_5^{x^2} \frac{\sin t}{t^4+9}\, dt$. Find $f'(x)$.

    Solution: Let $y = f(x)$. To use the Chain Rule, we may write $y = \displaystyle\int_5^{u} \frac{\sin t}{t^4+9}\, dt$, $u = x^2$, so $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

    Using the Fundamental Theorem of Calculus, we get $\dfrac{dy}{dx} = \dfrac{\sin u}{u^4+9} \cdot 2x$, so $f'(x) = \dfrac{2x\sin(x^2)}{x^8+9}$.