Mathematics 115 SOLUTIONS
Professor Alan H. Stein
Due Wednesday, March 14, 2007 This problem set is worth 50 points.

1. (a) Using plain language and avoiding all use of mathematical notation, state the *Product Rule.*

   Solution: The derivative of a product is equal to the first factor times the derivative of the second plus the second times the derivative of the first.

   (b) Using plain language and avoiding all use of mathematical notation, state the *Quotient Rule.*

   Solution: The derivative of a quotient is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

   (c) Using an appropriate blend of plain language and mathematical notation, state the *Chain Rule.*

   Solution: Given a composite function $y = f \circ g(x)$, if we write $y = f(u)$ with $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

   (d) Using plain language and avoiding all use of mathematical notation, write down a strategy for calculating derivatives that will always work for functions built up from basic, elementary functions.

   Solution: Differentiate term-by-term. For each term, determine whether the term is a basic elementary function, a product, a quotient or a composite function and then apply the appropriate rule.

2. Calculate $\frac{d}{dx}(x^3 \tan x - 5x + 3)$.

   Solution: $\frac{d}{dx}(x^3 \tan x - 5x + 3) = x^3 \sec^2 x + (\tan x) \cdot 3x^2 - 5 = x^3 \sec^2 x + 3x^2 \tan x - 5.$

3. Calculate $\frac{d}{dx}\left(\frac{x^3}{\sin x}\right)$.

   Solution: $\frac{d}{dx}\left(\frac{x^3}{\sin x}\right) = \frac{(\sin x)(3x^2) - x^3 \cos x}{\sin^2 x} = \frac{3x^2 \sin x - x^3 \cos x}{\sin^2 x}.$

4. Calculate $\frac{d}{dx}(\sin x \cos(x^4))$.

   Solution: $\frac{d}{dx}(\sin x \cos(x^4)) = (\sin x)\frac{d}{dx}(\cos(x^4)) + \cos(x^4) \cdot \cos x = \sin x(-\sin(x^4)) \cdot 4x^3 + \cos x \cos(x^4).$

5. Calculate $\dfrac{d}{dx}(\sin x \cos^4 x)$.

   Solution: $\dfrac{d}{dx}(\sin x \cos^4 x) = (\sin x)\dfrac{d}{dx}(\cos^4 x) + \cos^4 x \cos x = \sin x 4\cos^3 x(-\sin x) + \cos^5 x = \cos^5 x - 4\sin^2 x \cos^3 x$.

(6-7): Suppose the distance travelled along a straight path by an object is given by the formula $s = t^3 + 5t + \sin t$ for $t \geq 0$, where t represents time, measured in seconds, and s measures distance, measured in feet.

6. Find the average speed of the object over the time period $\pi \leq t \leq 2\pi$.

   Solution: Over the time period, the object goes a distance $(2\pi)^3 + 5 \cdot 2\pi + \sin 2\pi - [\pi^3 + 5\pi + \sin \pi] = 8\pi^3 + 10\pi + 0 - \pi^3 - 5\pi - 0 = 7\pi^3 + 5\pi$, so its average speed is $\dfrac{7\pi^3 + 5\pi}{\pi} = 7\pi^2 + 5$ feet per second.

7. Find the instantaneous speed of the object when $t = \pi$.

   Solution: Let v be the velocity. $v = \dfrac{ds}{dt} = 3t^2 + 5 + \cos t$, so $v|_{t=\pi} = 3\pi^2 + 5 + \cos \pi = 3\pi^2 + 5 - 1 = 3\pi^2 + 4$ and the instantaneous speed is $3\pi^2 + 5 - 1 = 3\pi^2 + 4$ feet per second.

8. Find an equation for the line tangent to the graph of the function $y = \tan x$ at the point $(\pi/6, 1/\sqrt{3})$.

   Solution: Since $y' = \sec^2 x$, we have $y'\big|_{x=\pi/6} = \sec^2(\pi/6) = (2/\sqrt{3})^2 = 4/3$. So the slope of the tangent is $4/3$ and its equation is $y - 1/\sqrt{3} = (4/3)(x - \pi/6)$.

9. Consider a function $y = f(x)$ defined implicitly by the equation $x = \sec y$, $y \in [0, \pi/2) \cup (\pi/2, pi]$. Find a formula for its derivative. *Extra Credit: Get the formula for its derivative in terms of x and state the common name for the function f.*

   Solution: *Note: The question was orginally given with a typo; the entire question will be graded on an extra credit basis for those who did it.*

   Differentiating implicitly, $\dfrac{d}{dx}(x) = \dfrac{d}{dx}(\sec y)$, so $1 = \sec y \tan y \dfrac{dy}{dx}$ and $\dfrac{dy}{dx} = \dfrac{1}{\sec y \tan y}$.

   Since $x = \sec y$, we have $1 + \tan^2 y = \sec^2 y$, so $1 + \tan^2 y = x^2$, $\tan^2 y = x^2 - 1$, $\tan y = \pm\sqrt{x^2 - 1}$ and thus $\dfrac{dy}{dx} = \pm\dfrac{1}{x\sqrt{x^2 - 1}}$.

   Since $\sec y \tan y = \dfrac{\sin y}{\cos^2 y} \geq 0$ for $y \in [0, \pi]$, we have $\dfrac{dy}{dx} = \dfrac{1}{|x|\sqrt{x^2 - 1}}$.

   The common name for the function is *arcsec*.

10. A 10 foot tall ladder is leaning against a vertical wall but the base of the ladder is slipping away at a rate of 2 feet per minute. How fast is the tip of the ladder slipping down when it is 7 feet from the floor?

    Solution: Let x be the distance from the base of the ladder to the wall and let y be the height of the tip of the ladder. We know $\frac{dx}{dt} = 2$ and we want to find $\frac{dy}{dt}|_{y=7}$.

    By the Pythagorean Theorem, we know $x^2 + y^2 = 100$, so $\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(100)$ and thus $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. It follows that $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$. Also, when $y = 7$, we have $x^2 + 7^2 = 100$, so $x^2 = 51$, $x = \sqrt{51}$. Plugging these in, we get $\sqrt{51} \cdot 2 + 7\frac{dy}{dt} = 0$, $\frac{dy}{dt} = -\frac{2\sqrt{51}}{7}$.

    So the tip of the ladder is slipping down at a rate of $\frac{2\sqrt{51}}{7}$ feet per minute.

11. *Extra Credit:* Find a positive real number δ such that $|\sqrt{x} - 5| < 0.01$ whenever $|x - 25| < \delta$.

    Solution: The following inequalities are equivalent:
    $|\sqrt{x} - 5| < 0.01$
    $|\sqrt{x} - 5||\sqrt{x} + 5| < 0.01|\sqrt{x} + 5|$
    $|x - 25| < 0.01|\sqrt{x} + 5|$.

    Since x will be close to 25, we may assume $x \geq 0$, in which case $|\sqrt{x} + 5| \geq 5$ and $0.01|\sqrt{x} + 5| \geq .05$.

    It follows that if $|x - 25| < 0.05$, the $|x - 25| < 0.05 \leq 0.01|\sqrt{x} + 5|$. We may thus take $\delta = 0.05$.