

Σ Sigma Notation

Sigma notation is a mathematical shorthand for expressing sums where every term is of the same form.

For example, suppose we want to write out the sum of all the integers from 1 to 100, inclusively. One might write

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20+21+22+23+24+25+26+27+28+29+30+31+32+33+34+35+36+37+38+39+40+41+42+43+44+45+46+47+48+49+50+51+52+53+54+55+56+57+58+59+60+61+62+63+64+65+66+67+68+69+70+71+72+73+74+75+76+77+78+79+80+81+82+83+84+85+86+87+88+89+90+91+92+93+94+95+96+97+98+99+100$$

This might strike one as being somewhat tedious. We might try writing something like $1+2+3+\dots+98+99+100$, leaving the middle terms to the imagination, or we might use Sigma Notation and simply write

$$\sum_{k=1}^{100} k,$$

which we may read as the sum, for k taking on every integer value starting with 1 and going up to 100, of all numbers of the form k .

Suppose we want to add together the squares of all the integers from 1 to 100. Again, we might write

$$1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2+11^2+12^2+13^2+14^2+15^2+16^2+17^2+18^2+19^2+20^2+21^2+22^2+23^2+24^2+25^2+26^2+27^2+28^2+29^2+30^2+31^2+32^2+33^2+34^2+35^2+36^2+37^2+38^2+39^2+40^2+41^2+42^2+43^2+44^2+45^2+46^2+47^2+48^2+49^2+50^2+51^2+52^2+53^2+54^2+55^2+56^2+57^2+58^2+59^2+60^2+61^2+62^2+63^2+64^2+65^2+66^2+67^2+68^2+69^2+70^2+71^2+72^2+73^2+74^2+75^2+76^2+77^2+78^2+79^2+80^2+81^2+82^2+83^2+84^2+85^2+86^2+87^2+88^2+89^2+90^2+91^2+92^2+93^2+94^2+95^2+96^2+97^2+98^2+99^2+100^2,$$

but it would be more convenient to write

$$1^2+2^2+3^2+\dots+98^2+99^2+100^2$$

or

$$\sum_{k=1}^{100} k^2.$$

We might write $\sum_{k=1}^5 (2k+1)$ rather than writing $3+5+7+9+11$.

In each of these examples, k is the *index*. The index does not have to be k . Other frequently used symbols are i , j , m and n .

Also, the index doesn't have to start with the value 1. As an example, we could have

$$\sum_{k=3}^6 (2k+1)$$

rather than writing $7+9+11+13$

or

$$\sum_{k=-5}^{-2} 5k$$

rather than writing $(-25) + (-20) + (-15) + (-10)$.

In general, we may interpret $\sum_{k=\alpha}^{\beta} a_k$ as the sum of all terms of the form a_k for all integer values of k between α and β .

Some Useful Formulas Involving Sums

$$\begin{aligned}\sum_{k=\alpha}^{\beta} ca_k &= c \sum_{k=\alpha}^{\beta} a_k \\ \sum_{k=\alpha}^{\beta} (a_k + b_k) &= \sum_{k=\alpha}^{\beta} a_k + \sum_{k=\alpha}^{\beta} b_k\end{aligned}$$

These two formulas are generalizations of the associative, commutative and distributive laws.

$$\begin{aligned}\sum_{k=1}^n 1 &= n \\ \sum_{k=1}^n k &= \frac{n(n+1)}{2} \\ \sum_{k=1}^n k^2 &= \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

These three formulas are useful in a number of calculations. We will demonstrate proofs of the second and third of these. The first should be obvious.

We will prove the second formula two different ways. The first comes from simply writing down the sum two different ways, frontwards and backwards.

Proof. Let $S = \sum_{k=1}^n k$. We may write $S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$. Writing

$$\begin{array}{cccccccc} 2S = S + S = & 1 & + & 2 & + & \cdots + & n-1 & + & n & + \\ & n & + & n-1 & + & \cdots + & 2 & + & 1 & \end{array}$$

Adding all the terms by adding together the pairs which lie above and below each other, we get $2S = (n+1) + (n+1) + \cdots + (n+1) + (n+1) = n(n+1)$, from which it follows that $S = \frac{n(n+1)}{2}$. \square

The second proof will use a very useful method called *Mathematical Induction*.

Mathematical Induction is often used to prove that statements involving an arbitrary integer n are true for all positive integral values of that integer. It works as follows.

We may have a proposition we may denote by $P(n)$, where n represents an arbitrary integer. Using *Mathematical Induction*, we need to prove two statements:

- (1) $P(1)$ is true.
- (2) If $P(n)$ is true, then $P(n+1)$ is also true.

If we can prove both statements, then $P(n)$ must be true for all positive integers n .

Proof. In this instance, the assertion $P(n)$ corresponds to the assertion $\sum_{k=1}^n k = \frac{n(n+1)}{2}$ and $P(n+1)$ corresponds to the assertion $\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$.

Thus, $P(1)$ corresponds to the assertion $\sum_{k=1}^1 k = \frac{1(1+1)}{2}$.

But $\sum_{k=1}^1 k = 1$, while $\frac{1(1+1)}{2} = \frac{2}{2} = 1$, so $P(1)$ is certainly true.

Now assume $P(n)$ is true. This means $\sum_{k=1}^n k = \frac{n(n+1)}{2}$.

It follows that $\sum_{k=1}^{n+1} k = \sum_{k=1}^n k + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1) \left(\frac{n}{2} + 1 \right) = (n+1) \cdot \frac{n+2}{2} = \frac{(n+1)(n+2)}{2}$, so $P(n+1)$ is true. \square

We will also prove the third formula two different ways. The first proof will involve Mathematical Induction, while the second proof will involve an interesting algebraic trick that can be generalized to prove similar formulas for summing higher powers.

Proof. We wish to prove

$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all positive integer values of n .

If $n = 1$, this corresponds to the statement

$\sum_{k=1}^1 k^2 = \frac{1 \cdot 2 \cdot 3}{6}$, which is obviously true.

Now suppose the assertion is true for a given value of n . It then follows that

$$\begin{aligned} \sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^n k^2 + (n+1)^2 = \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \\ &= \frac{n+1}{6} \cdot (n(2n+1) + 6(n+1)) = \\ &= \frac{n+1}{6} \cdot (2n^2 + n + 6n + 6) = \\ &= \frac{n+1}{6} \cdot (2n^2 + 7n + 6) = \\ &= \frac{n+1}{6} \cdot (n+2)(2n+3) = \\ &= \frac{[n+1]([n+1]+1)(2[n+1]+1)}{6}. \end{aligned}$$

\square

The next proof involves the interesting algebraic trick.

Proof. We start with the following calculation:

$$\text{Let } S = [1 \cdot 2 \cdot 3 - 0 \cdot 1 \cdot 2] + [2 \cdot 3 \cdot 4 - 1 \cdot 2 \cdot 3] + [3 \cdot 4 \cdot 5 - 2 \cdot 3 \cdot 4] + \dots + [n \cdot (n+1) \cdot (n+2) - (n-1) \cdot n \cdot (n+1)].$$

On the one hand, this is a *telescoping sum*, with a lot of cancellation, leaving the conclusion $S = n(n+1)(n+2)$ after all the cancellation.

On the other hand, we can write $S = 3(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n \cdot (n+1))$.

Looking at the two representations of S , we can conclude

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}.$$

In Sigma Notation, this may be written

$$\sum_{k=1}^n k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

We may then observe

$$\begin{aligned} \sum_{k=1}^n k^2 &= \sum_{k=1}^n k \cdot k = \\ \sum_{k=1}^n [k(k+1) - k] &= \\ \sum_{k=1}^n k(k+1) - \sum_{k=1}^n k &= \\ \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} &= \\ \frac{n(n+1)}{6} \cdot (2(n+2) - 3) &= \\ \frac{n(n+1)}{6} \cdot (2n+1) &= \\ \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

□

Riemann Sums

Definition 1 (Partition). *A partition of an interval $[a, b]$ with $a \leq b$ is a set $\{x_0, x_1, x_2, \dots, x_n\}$ where $a = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n = b$.*

Definition 2 (Riemann Sum). *Let f be a function defined on $[a, b]$ and let $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Let $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, 3, \dots, n$ and let $c_k \in [x_{k-1}, x_k]$ for $k = 1, 2, 3, \dots, n$. We define the Riemann Sum for f for the partition \mathcal{P} on the interval $[a, b]$ as $\mathcal{R}(f, \mathcal{P}, a, b) = \sum_{k=1}^n f(c_k) \Delta x_k$.*

If f is a nice, positive, continuous function then $\mathcal{R}(f, \mathcal{P}, a, b)$ may be interpreted as a sum of areas of rectangles, giving an approximation to the area bounded by the graph of f , the x -axis and the lines $x = a$, $x = b$.

We will sometimes write $\mathcal{R}(f, \mathcal{P})$ or even just $\mathcal{R}(f)$ instead of $\mathcal{R}(f, \mathcal{P}, a, b)$.

The Definite Integral

As the widths Δx_k of the subintervals approach 0, the Riemann Sums *hopefully* approach a limit. If that happens, we call the limit the definite integral of f from a to b and denote it by $\int_a^b f(x) dx$.

We immediately get the application that if f is positive and continuous, then $\int_a^b f(x) dx$ is equal to the area of the region bounded by the graph of f , the x -axis and the lines $x = a$, $x = b$.

Calculating Definite Integrals The Fundamental Theorem of Calculus

Theorem 1 (FTC-Part I). *If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t) dt$ is defined on $[a, b]$ and $F'(x) = f(x)$.*

Theorem 2 (FTC-Part II). *If f is continuous on $[a, b]$ and $F(x) = \int f(x) dx$ on $[a, b]$, then $\int_a^b f(x) dx = F(x)|_a^b = F(b) - F(a)$.*

Note the introduction of a notation:

$$F(x)|_a^b = F(b) - F(a)$$

Example

$$\int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

Example

$$\int_0^\pi \sin x dx = (-\cos x) \Big|_0^\pi = -\cos \pi - (-\cos 0) = -(-1) - (-1) = 2.$$

Example

Calculate $\int_0^1 \frac{x}{x^2 + 1} dx$.

This will require slightly more work, since it's a little harder to find an antiderivative in this case. We'll use the method of substitution.

Let $I = \int \frac{x}{x^2 + 1} dx$ and let $u = x^2 + 1$.

$$\frac{du}{dx} = 2x$$

$$du = 2x dx$$

$$dx = \frac{du}{2x}$$

$$I = \int \frac{x}{u} \frac{du}{2x} = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| = \frac{1}{2} \ln(x^2 + 1) + c$$

$$\text{Thus } \int_0^1 \frac{x}{x^2 + 1} dx = \frac{1}{2} \ln(x^2 + 1) \Big|_0^1 = \frac{1}{2} \ln(1^2 + 1) - \frac{1}{2} \ln(0^2 + 1) = \frac{1}{2} \ln 2 - \frac{1}{2} \ln 1 = \frac{1}{2} \ln 2.$$