Antiderivatives

Definition 1 (Antiderivative). *If $F'(x) = f(x)$ we call F an antiderivative of f.*

Definition 2 (Indefinite Integral). *If F is an antiderivative of f, then $\int f(x)\,dx = F(x) + c$ is called the (general) Indefinite Integral of f, where c is an arbitrary constant.*

The indefinite integral of a function represents every possible antiderivative, since it has been shown that if two functions have the same derivative on an interval then they differ by a constant on that interval.

Terminology: When we write $\int f(x)\,dx$, $f(x)$ is referred to as the *integrand*.

Basic Integration Formulas

As with differentiation, there are two types of formulas, formulas for the integrals of specific functions and structural type formulas. Each formula for the derivative of a specific function corresponds to a formula for the derivative of an elementary function. The following table lists integration formulas side by side with the corresponding differentiation formulas.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \text{ if } n \neq -1 \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

$$\int \sin x\,dx = -\cos x + c \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\int \cos x\,dx = \sin x + c \qquad \frac{d}{dx}(\sin x) = \cos x$$

$$\int \sec^2 x\,dx = \tan x + c \qquad \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\int e^x\,dx = e^x + c \qquad \frac{d}{dx}(e^x) = e^x$$

$$\int \frac{1}{x}\,dx = \ln x + c \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\int k\,dx = kx + c \qquad \frac{d}{dx}(kx) = k$$

Structural Type Formulas

We may integrate *term-by-term*:

$\int kf(x)\,dx = k\int f(x)\,dx$

$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

In plain language, the integral of a constant times a function equals the constant times the derivative of the function and the derivative of a sum or difference is equal to the sum or difference of the derivatives.

These formulas come straight from the corresponding formulas for calculating derivatives and are used the same way.

Integrating Individual Terms

When calculating derivatives of individual terms, one needs to recognize whether the term is an elementary function, a product, a quotient or a composite function. There is a little bit more art to integration, at least if the term is not the derivative of an elementary function.

Integration is essentially the reverse of differentiation, so one might expect formulas for reversing the effects of the Product Rule, Quotient Rule and Chain Rule. This is almost the case. There is a formula, called the Integration By Parts Formula, for reversing the effect of the Product Rule and there is a technique, called Substitution, for reversing the effect of the Chain Rule. There is no specific formula or technique for reversing the effect of the Quotient Rule, but one is not really necessary since the Quotient Rule is redundant.

Integration also becomes an art because not only isn't it always obvious whether one should resort to Integration By Parts or the Substitution Technique but the use of the Integration By Parts Formula and the Substitution Technique is not as straightforward as the use of the Product, Quotient or Chain Rule.

The Substitution Technique

The substitution technique may be divided into the following steps. Every step but the first is purely mechanical. With a little bit of practice (in other words, make sure you do the homework problems assigned), you should have no more difficulty carrying out a substitution than you should be having by now when you differentiate.

Note: In the following, we will assume that you are trying to calculate an integral $\int f(x)dx$. If the dummy variable is called something other than x, then some of the names you would use for variables might be different.

(1) Choose a substitution $u = g(x)$.

Some suggestions on how to choose a substitution will be made later.

(2) Calculate the derivative $\frac{du}{dx} = g'(x)$.

(3) Treating the derivative *as if it were a fraction*, solve for dx:

$$\frac{du}{dx} = g'(x), \quad du = g'(x)dx, \quad dx = \frac{du}{g'(x)}.$$

(4) Go back to the original integral and replace $g(x)$ by u and replace dx by $\frac{du}{g'(x)}$.

(5) Simplify.

Every incidence of x should cancel out at this step. If not, you will need to try another substitution. Make sure that you simplify properly—this is the easiest step to make mistakes during.

(6) Integrate.

(7) Replace u by $g(x)$ in your result.

(8) Check your answer (of course).

Choosing an Appropriate Substitution

This is the only non-routine part of carrying out a substitution, but should not be at all difficult for any student who has mastered the art of differentiation. There are two basic tactics for choosing a substitution. Each will work in the vast majority of cases where a routine substitution is needed. Since neither will work in all cases, you need to be comfortable with both. Therefore, you should try using both methods on the same problem wherever possible. (There are quite a few non-routine substitutions that are used in special situations. They need to be learned separately.)

The First Method

The method most students probably find easiest to use relies on familiarity with the chain rule for differentiation. In order to decide on a useful substitution, look at the integrand and pretend that you are going to calculate its derivative rather than its integral. (You usually don't actually have to write anything down—you can usually just visualize the steps.) Look to see if there is any step during which you would have to use the chain rule. If so, think of the decomposition you would have to make, i.e. the step where you would write down something like $y = f(u), \quad u = g(x)$. Try the substitution $u = g(x)$.

The Second Method

This method involves looking at parts of the integrand and observing whether the derivative of part of the integrand equals some other factor of the integrand. If so, u may be substituted for that part. (In deciding, you may ignore constant factors, since they are easy to manipulate around.)