# Derivatives of Logarithmic and Exponential Functions

*We will ultimately go through a far more elegant development then what we can do now.*

Consider first an exponential function of the form $f(x) = a^x$ for some constant $a > 0$.

*Note the difference between a power function $x \to x^n$ and an exponential function $x \to a^x$. For a power function, the variable is raised to a power; for an exponential function, a constant is raised to a variable power.*

Let's try to calculate the derivative for $f$.

Using the definition of a derivative, we may write

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$
$$\lim_{h \to 0} \frac{a^{x+h} - a^x}{h} =$$
$$\lim_{h \to 0} \frac{a^x(a^h - 1)}{h} =$$
$$a^x \lim_{h \to 0} \frac{a^h - 1}{h}.$$

We may write $\frac{d}{dx}(a^x) = k \cdot a^x$, where $k = \lim_{h \to 0} \frac{a^h - 1}{h}$ depends on $a$.

*We are assuming $k$ exists! It does, but this is not so easy to show.*

Evaluating $\frac{2^h - 1}{h}$ for values of $h$ close to 0 yields values close to 0.69, while evaluating $\frac{3^h - 1}{h}$ for values of $h$ close to 0 yields values close to 1.1.

$k \cdot a^x$ is obviously simplest if $k = 1$. The numerical calculations suggest there is some $2 < a < 3$ for which $a = 1$. That number is called $e$, yielding the formula

$$\frac{d}{dx}(e^x) = e^x.$$

# The Exponential Function

The function $x \to e^x$ is called *the exponential function* and is often denoted by exp.

The exponential function is essentially unique in having the property that it's its own derivative!

*The adjective essentially is used because every constant multiple of the exponential function has the same property, but no other function has that property!*

If we interpret the derivative as a measure of rate of change, the fact that the exponential function is its own derivative may be interpreted to mean that the rate at which the exponential function changes is equal to the magnitude of the exponential function.

It turns out that all functions whose rates of change are proportional to their sizes are exponential functions. *Note the omission of the definite article.*

## The Natural Logarithm Function

Recall the definition of a logarithm function:

$\log_b x$ is the power which $b$ must be raised to in order to obtain $x$. In other words, $l = \log_b x$ if $b^l = x$.

The logarithm with base $e$ is known as the *natural logarithm function* and is denoted by ln. Thus, $l = \ln x$ if and only $e^l = x$.

We'll try to figure out the derivative of the natural logarithm function ln. Our calculations will not be rigorous; we will obtain the correct formula, but a legitimate derivation will have to wait until we learn about the *definite integral*.

Let $f(x) = \ln x$. Let's start calculating $f'(x)$.

According to the definition of a derivative, $f'(x) = \lim_{z \to x} \dfrac{f(z) - f(x)}{z - x} = \lim_{z \to x} \dfrac{\ln z - \ln x}{z - x}$.

We need to estimate the difference quotient $\dfrac{\ln z - \ln x}{z - x}$ when $z$ is close to $x$. We'll do it in a rather strange way.

Let $Z = \ln z$ and $X = \ln x$. Then we know $z = e^Z$ and $x = e^X$ and we may write $\dfrac{\ln z - \ln x}{z - x}$ as $\dfrac{Z - X}{e^Z - e^X}$.

Now, let's go back and take another look at the derivative of the exponential function, but from a different perspective and with slightly different notation. *Sometimes it pays to write something a few different ways!*

Let $g(X) = e^X$. By the definition of a derivative,

$g'(X) = \lim_{Z \to X} \dfrac{e^Z - e^X}{Z - X}$.

But we know $g'(X) = e^X$, so this suggests that when $Z$ is close to $X$, $\dfrac{e^Z - e^X}{Z - X}$ is close to $e^X$.

But $\dfrac{e^Z - e^X}{Z - X}$ is the reciprocal of $\dfrac{Z - X}{e^Z - e^X}$, suggesting that $\dfrac{Z - X}{e^Z - e^X}$ is close to $\dfrac{1}{e^X}$.

On the other hand, when trying to find the derivative of the natural log function we came up something suggesting $\dfrac{Z - X}{e^Z - e^X}$ was close to the derivative. This suggests that the derivative is equal to $\dfrac{1}{e^X}$.

Recall $e^X = x$, so $\dfrac{1}{e^X} = \dfrac{1}{x}$, which suggests

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Indeed, this is the derivative, although we're not ready for a rigorous derivation. We will, however, make use of this formula.

## Logarithms to Other Bases

The key properties of logarithms are:

$\log_b(xy) = \log_b x + \log_b y$ (The log of a product equals the sum of the logs.)

$\log_b(x/y) = \log_b x - \log_b y$ (The log of a quotient equals the difference of the logs.)

$\log_b(x^r) = r \log_b x$ (The log of something to a power is the power times the log.)

We can use these properties to show that, in a very real sense, natural logarithms suffice and we can always write any logarithm in terms of a natural logarithm.

Suppose $l = \log_b x$. It follows that $b^l = x$ and thus $\ln(b^l) = \ln x$.

Using the third property of logarithms, we see $l \ln b = \ln x$ and, solving for $l$, we get $l = \dfrac{\ln x}{\ln b}$. This gives us the crucial identity

$$\log_b x = \frac{\ln x}{\ln b}.$$

This enables us to calculate derivatives involving logarithms to any base, as shown in the following general example.

## Example

Find $\frac{d}{dx}(\log_b x)$.

Solution: $\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b}\frac{d}{dx}(\ln x) = \frac{1}{\ln b} \cdot \frac{1}{x} = \frac{1}{x \ln b}$.

## Other Exponential Functions

A calculation similar to the derivation of the identity $\log_b x = \frac{\ln x}{\ln b}$ yields a useful identity involving exponential functions.

Let $a^x = y$. Then:

$\ln(a^x) = \ln y$

$x \ln a = \ln y$

$e^{x \ln a} = e^{\ln y}$.

Since $e^{\ln y} = y$ and $y = a^x$, it follows that

$a^x = e^{x \ln a}$.

## Caution:

In high school algebra, a meaning was given to rational exponents: If $a > 0$, $m, n \in \mathbb{Z}$, $n > 0$, then $a^{m/n} = \sqrt[n]{a^m}$. However, no meaning was given to $a^x$ if $x$ is irrational. That can be done and the identity $a^x = e^{x \ln a}$ will play a key role.

Let's take another look at the derivative of ordinary exponential functions. We found

$\frac{d}{dx}(a^x) = k \cdot a^x$, where $k = \lim_{h \to 0} \frac{a^h - 1}{h}$.

Let's play the same sort of game we played when trying to calculate the derivative of the natural log function and let $a^h = H$, noting that $\frac{a^h - 1}{h}$ will be close to $k$ when $h$ is close to 0, and hence when $H$ is close to 1.

$\frac{a^h - 1}{h}$ may be written as $\frac{H - 1}{\log_a H - \log_a 1}$, since $\log_a 1 = 0$.

If we let $f(x) = \log_a x$ and tried to calculate $f'(1)$, we might write

$f'(1) = \lim_{H \to 1} \frac{f(H) - f(1)}{H - 1} =$

$\lim_{H \to 1} \frac{\log_a H - \log_a 1}{H - 1}$.

But we earlier showed that $f'(x) = \frac{1}{x \ln a}$, so $f'(1) = \frac{1}{\ln a}$.

This suggests that $\dfrac{\log_a H - \log_a 1}{H - 1}$ is close to $\dfrac{1}{\ln a}$ if $H$ is close to 1, which suggests its reciprocal $\dfrac{H - 1}{\log_a H - \log_a 1}$, and hence $\dfrac{a^h - 1}{h}$ as well, is close to $\ln a$ if $H$ is close to 1 and $h$ is close to 0.

We thus expect $k = \ln a$ and $\dfrac{d}{dx}(a^x) = a^x \ln a$.