Derivatives of Trigonometric Functions

Let $f(x) = \sin x$. From the definition of a derivative,

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} =$

$\lim_{h\to 0} \dfrac{\sin(x+h) - \sin x}{h}$.

Conveniently, we have a trigonometric identity that enables us to rewrite $\sin(x+h)$ as $\sin x \cos h + \cos x \sin h$, so we have

$f'(x) = \lim_{h\to 0} \dfrac{\sin x \cos h + \cos x \sin h - \sin x}{h} =$

$\lim_{h\to 0} \dfrac{\sin x \cos h - \sin x + \cos x \sin h}{h} =$

$\lim_{h\to 0} \dfrac{\sin x(\cos h - 1) + \sin h \cos x}{h} =$

$\lim_{h\to 0} \left(\sin x \dfrac{\cos h - 1}{h} + \cos x \dfrac{\sin h}{h} \right) =$

$\sin x \lim_{h\to 0} \dfrac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \dfrac{\sin h}{h}$.

Two Important Limits

We will show that $\lim_{h\to 0} \dfrac{\sin h}{h} = 1$ and

$\lim_{h\to 0} \dfrac{1 - \cos h}{h} = 0$, from which it will follow that

$f'(x) = (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x$.

We thus have the formula $\dfrac{d}{dx}(\sin x) = \cos x$ subject to proving the claims about the limits of $\dfrac{\sin h}{h}$ and $\dfrac{1 - \cos h}{h}$.

$\lim_{h\to 0} \dfrac{\sin h}{h} = 1$

Claim 1. $\lim_{h\to 0} \dfrac{\sin h}{h} = 1$.

Proof: First consider $0 < h < \pi/2$, draw the unit circle with center at the origin, and consider the sector with central angle h where one side lies along the x-axis and the other side lies in the first quadrant. Since the area of the circle is π and the ratio of the area of the sector to the area of the circle is $\dfrac{h}{2\pi}$, the area of the sector is $\dfrac{h}{2\pi} \cdot \pi = \dfrac{h}{2}$.

Now consider the right triangle where the hypotenuse coincides with the side of the sector lying in the first quadrant and the base lies along the x-axis. The vertices will be $(0, 0)$, $(\cos h, 0)$, $(\cos h, \sin h)$, so its legs will be of length $\cos h$, $\sin h$ and its area will be $\frac{1}{2} \cdot \cos h \sin h$.

Since the triangle is contained within the sector, its area will be smaller than the area of the sector. Hence $\frac{1}{2} \cdot \cos h \sin h < \frac{h}{2}$.

Multiplying both sides by $\frac{2}{h \cos h}$ yields the inequality $\frac{\sin h}{h} < \frac{1}{\cos h}$.

Now consider the right triangle with one leg coinciding with the side of the sector lying along the x-axis and the hypotenuse making an angle h with that leg. Its vertices are $(0, 0)$, $(1, 0)$, $(1, \tan h)$, so its legs will be of length 1, $\tan h$ and its area will be $\frac{1}{2} \cdot \tan h$.

Since the sector is contained within this triangle, its area will be smaller than the area of the triangle. Hence $\frac{h}{2} < \frac{1}{2} \cdot \tan h$.

Multiplying both sides by $\frac{2 \cos h}{h}$ and making use of the identity $\tan h \cos h = \sin h$ yields the inequality $\cos h < \frac{\sin h}{h}$.

Combining the two inequalities we have obtained yields

$$\cos h < \frac{\sin h}{h} < \frac{1}{\cos h} \tag{1}$$

if $0 < h < \pi/2$.

Now, suppose $-\pi/2 < h < 0$. Then $0 < -h < \pi/2$ and the double inequality (1) yields

$$\cos(-h) < \frac{\sin(-h)}{-h} < \frac{1}{\cos(-h)}. \tag{2}$$

Since $\cos(-h) = \cos h$ and $\sin(-h) = -\sin h$, it follows that $\frac{\sin(-h)}{-h} = \frac{-\sin h}{-h} = \frac{\sin h}{h}$ and (2) becomes

$$\cos h < \frac{\sin h}{h} < \frac{1}{\cos h}. \tag{3}$$

We thus see (1) holds both for $0 < h < \pi/2$ and for $-\pi/2 < h < 0$.

Since $\lim_{h\to 0} \cos h = \lim_{h\to 0} \dfrac{1}{\cos h} = 1$, by the *Squeeze Theorem* it follows that $\lim_{h\to 0} \dfrac{\sin h}{h} = 1$ QED

Claim 2. $\lim_{h\to 0} \dfrac{1-\cos h}{h} = 0$.

We make use of the identity involving sin and an algebraic manipulation reminiscent of rationalization, enabling us to prove the claim with a fairly routine calculation.

Proof

Proof. $\lim_{h\to 0} \dfrac{1-\cos h}{h} = \lim_{h\to 0} \dfrac{1-\cos h}{h} \cdot \dfrac{1+\cos h}{1+\cos h} =$
$\lim_{h\to 0} \dfrac{1-\cos^2 h}{h(1+\cos h)} = \lim_{h\to 0} \dfrac{\sin^2 h}{h(1+\cos h)} =$
$\lim_{h\to 0} \dfrac{\sin h}{h} \cdot \dfrac{\sin h}{1+\cos h} =$
$\lim_{h\to 0} \dfrac{\sin h}{h} \cdot \lim_{h\to 0} \dfrac{\sin h}{1+\cos h} = 1 \cdot 0 = 0.$ □